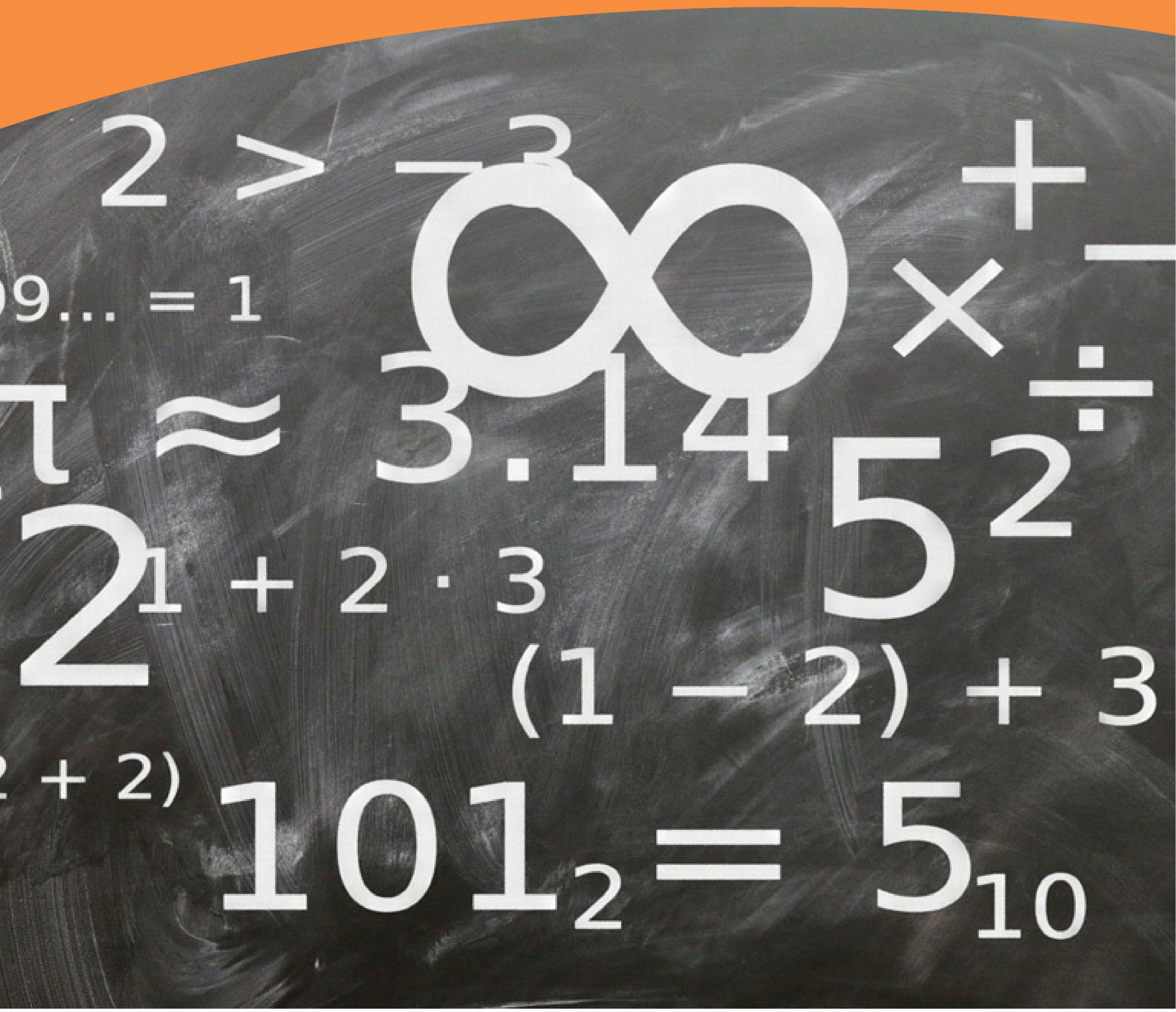
Essential Mathematics for Engineers

W. J. R. H. Pooler



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W J R H POOLER

ESSENTIAL MATHEMATICS FOR ENGINEERS

Essential Mathematics for Engineers
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CONTENTS

ABOUT THE AUTHOR

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ONC, MA (Cantab) class 1, CENG, MIEE, MIMechE

I studied for and obtained Ordinary National Certificate while working as an apprentice at the English Electric Co, Stafford. This included time in their high voltage laboratory. I then went to Cambridge University and passed the Mechanical Sciences Tripos after two years with First Class Honours. For the third year, I carried out further studies on heavy electrical power machines. After graduating, I joined the Iraq Petroleum Company in Kirkuk, Iraq and was later appointed Protection Engineer and System Control Engineer responsible for the operation of the high voltage network and for the hands on commissioning of all new electrical plant including 66kv and 11kv cables and lines and transformers up to 5MVA and motors up to 2000 hp.

I was then appointed Head of Electrical Engineering at Basrah Petroleum Co responsible for four power stations, 33kv and 11kv transformers, cables and lines and motors up to 1500 hp. The operation of all Instrumentation in the Production Plants and all Telecommunications in the Company was later added to my responsibilities.

This book is based on experiences gained during this period.

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FOREWORD

This book is a record of mathematics notes made while at school, at university and after university.

The book is an aide memoir or reference book rather than a textbook. The book begins with a Summary to help in its use as an aide memoir. It is hoped that this will allow quick and easy access to the main text where further study is required. The Summary of each topic begins with definitions to help with the jargon. Some words have a special meaning in mathematics. The main body of the text is developed from first principles so that nothing has to be taken on trust. (An exception is Taylor's theorem but this has limited application and is not used again in this document).

The notes are arranged so that the minimum amount of information need be committed to memory. In the Summary, items that could be memorised are coloured in red. Other results follow easily from these. In the main text, all significant results are coloured red to highlight them for easy reference whether or not they should be memorised.

There are often several ways to tackle any problem. It is often not clear which way is best and which ways lead to a dead ends.

The solution of Bessells equations has been covered in some depth. However it is usually sufficient to just recognise the equation and write down the answer.

The main text contains many examples. These are included as they demonstrate how the various results are applied to solve actual problems. The examples are in my notes. I do not know where they came from.

John Pooler

SUMMARY (PART 1 PURE)

Arithmetic

Definitions;
Sum = One number plus another
Difference = One number minus another
Product = One number times another
Quotient = One number divided by another
A number is the Product of its Factors
Primes are Numbers with no Factors except 1 and itself
HCF (Highest Common Factor) = Highest Factor that is Common to all numbers of a group
LCM (Lowest Common Multiplier) = Lowest number that has all numbers of a group as Factors
Numerator is the Number at top of Fraction
Denominator is the Number at bottom of Fraction (Down below)
Reciprocal = 1 Divided by the Number
Factorial is the Product of all Numbers from 1 to the Number and is written with !
For example 4! = 1 x 2 x 3 x 4
Ratio is the Comparison of 2 or more Numbers. For example 15 : 5 has the same Ratio as 3 : 1
Square of a Number = Number times itself, written as N^2. For example $5^2 = 25$
Square Root of a Number times itself = The Number. Square Root is written as $\sqrt{N}$.
For example $\sqrt{25} = \pm 5$
Index, or Power = Number of times a Number is multiplied by itself.
For example $5^3 = 5 \times 5 \times 5$ has the index of 3
Scientific Notation = Number expressed as a number between 0 and 10 times powers of 10
Binary = Number expressed in 2 digits (0 & 1)
Octal = Number expressed in 8 digits (0 - 7)
Hexadecimal = Number expressed in 16 digits (0 - 9 and A - F)

Hex(abcd) = Decimal $(d + c \times 16 + b \times 16^2 + a \times 16^3)$
$2^{10} = 1024$ $\sqrt{2} \approx \pm 1.414$ $1/\sqrt{2} \approx \pm 0.707$ $\sqrt{3} \approx \pm 1.732$ $\sqrt{10} \approx \pm 3.16$

Algebra

Definitions;
Coefficient and Constant Term..
For example in the function $7x^2 - 5x + 3$, the Coefficient of x^2 is 7
and the Coefficient of x is (-5) and the Constant term is 3
Equations are statements that two functions are equal
Simultaneous equations are a set of Equations connecting two or more unknowns

Irrational Functions are functions that contain a square root, or cube root etc.
Rationalised Functions do not contain a square root, or cube root etc.
Greek letter sigma $\sum$ means "Sum of Terms Like"

$(-a)$ times $(-b) = +ab$
a^m times $a^n = a^{m+n}$
$(a^m)^n = a^{mn}$
$a^0 = 1$ and $a^1 = a$ and $a^{-n} = 1/a^n$ and $a^{(1/n)} = {}^n\sqrt{(a)}$
$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$
To factorize $ax^2 + bx + c$. If ac (ie a times c) is negative, look for factors of ac whose sum = b
If ac is positive, look for factors whose difference = $\pm b$.
For example $10x^2 + x - 3$. $ac = -30$. Factors are 6 and -5
$10x^2 + x - 3 = (5x + 3)(2x - 1)$
$x^2 - a^2 = (x + a)(x - a)$
$(x \pm a)$ is a factor of $x^3 \pm a^3$ Put $a = 1$ to get $(x \pm 1)$ is a factor of $x^3 \pm 1$
$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ can be divided by $b_0 + b_1x + b_2x^2 + \ldots + b_{n-r}x^{n-r}$ to get the Quotient and Remainder. Method is similar to Arithmetical Long Division.
Include missing terms using zero as the coefficient.
For example $(ax^2 + bx + c)/(x + d) = [ax + bd - ad] + [(c - bd + ad^2)/(x + d)]$
Divide $F(x)$ by $(x - a)$ and the Remainder is $F(a)$
$F(x) / [(x + a_1)(x + a_2)(x + a_3)] = A_1/(x + a_1) + A_2/(x + a_2) + A_3/(x + a_3)$
where $A_1 = F(-a_1) / [(a_2 - a_1)(a_3 - a_1)]$ etc
If the Numerator is the same or higher power than the Denominator, then first divide the Numerator by the Denominator.
In each fraction, the Numerator contains x to one power less than the Denominator
Two equal factors $F(x) / [(x + a_1)^2(x + a_2)] = A_1/(x + a_1) + A_2/(x + a_1)^2 + A_3/(x + a_2)$
$1/(a + \sqrt{b}) = (a - \sqrt{b})/(a^2 - b)$ and $1/(a - \sqrt{b}) = (a + \sqrt{b})/(a^2 - b)$
These put the irrational term in the numerator
$i^2 = -1$
$(a + ib)(a - ib) = a^2 + b^2$
Therefore $1/(a + ib) = (a - ib)/(a^2 + b^2)$ and $1/(a - ib) = (a + ib)/(a^2 + b^2)$.
These put the complex term in the Numerator.
Solution to the quadratic $Ax^2 + Bx + C = 0$ is $x = [-B \pm \sqrt{(B^2 - 4AC)}] / 2A$
$\alpha_1 + \alpha_2 = -B/A$ and $\alpha_1 \alpha_2 = C/A$ where α_1 are α_2 are the two solutions
If $4AC > B^2$ then the two solutions are a conjugate pair, $\alpha + i\beta$ and $\alpha - i\beta$
An equation has as many solutions as the highest power of x after rationalizing.
A quadratic has 2 solutions, a cubic has 3.

Geometry

Definitions;
Angles, one revolution is 360 degrees = 2 πradians, a Right Angle is 90 degrees = $\pi/2$ radians
Equilateral Triangle has all sides equal
Isosceles Triangle has two angles equal
Similar Triangles are Two Triangles with same angles.
The sides are in same ratio in both triangles.
Congruent Triangles are Two Triangles exactly the same
"Normal to", "Orthogonal to" and "Perpendicular to" mean "at right angles to"
Hypotenuse is the Side of a Triangle opposite a right angle
Tangent is a line that just touches a curve and is parallel to the curve at that point.
Definition of radian. Angle in radians = Length of arc of a circle divided by radius.
For a circle, the length of arc is θtimes the radius $s = r\,\theta$
Conditions for congruent triangles.
Same on both either (i) 3 sides or (ii) 2 sides and the angle between them or (iii) 2 angles and a corresponding side or (iv) hypotenuse and one other side.
Triangle Sum of angles = 180^0 = πradians
Area = (½) Base x Height = (½) ab SinC = $\sqrt{[s(s-a)(s-b)(s-c)]}$ where $s = (½)(a + b + c)$
a/SinA = b/SinB = c/SinC and $c^2 = a^2 + b^2 - 2ab\text{Cos}C$
Pythagoras (for a Right Angled Triangle) $a^2 + b^2 = c^2$
Examples $3^2 + 4^2 = 5^2$, $6^2 + 8^2 = 10^2$, $12^2 + 5^2 = 13^2$
Medians meet at a point, so do Angle bisectors, so do lines from each apex perpendicular to opposite side, so do perpendiculars from mid points of sides
Circle. Circumference = $\pi D = 2\pi R$ and Area = πR^2 where D is the Diameter and R the Radius
Rectangle or Parallelogram Area = Base x Height (Height is measured normal to base)

Trigonometry

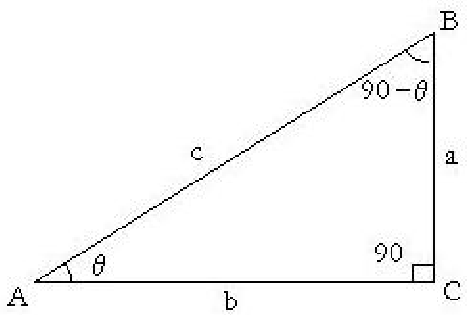


Figure 1: Sides and angles of a Triangle

Sinθ = a/c and Cosθ = b/c and Tanθ = a/b
Cosecθ = c/a = 1/ Sinθ and Secθ = c/b = 1/Cosθ and Cotθ = b/a = 1/ Tanθ

Sinθ/Cosθ = Tanθ

$\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$
$\sin(90^0 - \theta) = \cos\theta$ and $\tan(90^0 - \theta) = \cot\theta$
$\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = +\cos\theta$ and $\tan(-\theta) = -\tan\theta$
$\sin(180^0 - \theta) = +\sin\theta$ and $\cos(180^0 - \theta) = -\cos\theta$ and $\tan(180^0 - \theta) = -\tan\theta$
$\sin(180^0 + \theta) = -\sin\theta$ and $\cos(180^0 + \theta) = -\cos\theta$ and $\tan(180^0 + \theta) = +\tan\theta$

CAST Angles $(-90^0$ to $0^0)$ **C**os + ive, Sin and Tan – ive
Angles $(0^0$ to $90^0)$ **A**ll + ive
Angles $(90^0$ to $180^0)$ **S**in + ive, Cos and Tan – ive
Angles $(180^0$ to $270^0)$ **T**an + ive, Sin and Cos – ive

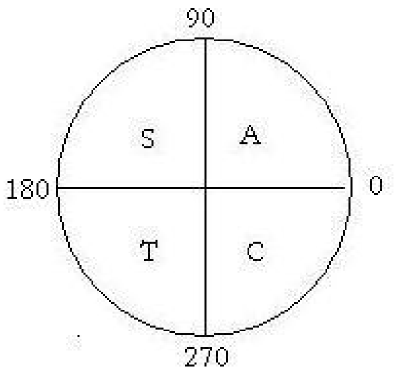


Figure 2: CAST

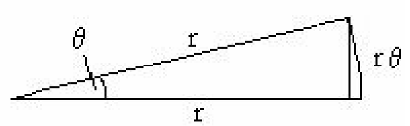


Figure 3: Small angles

If θ is small and in radians then $\sin\theta = \theta$ and $\tan\theta = \theta$ and $\cos\theta = 1 - (\frac{1}{2})\theta^2$

For Sin, Cos and Tan of other angles see Figure 4

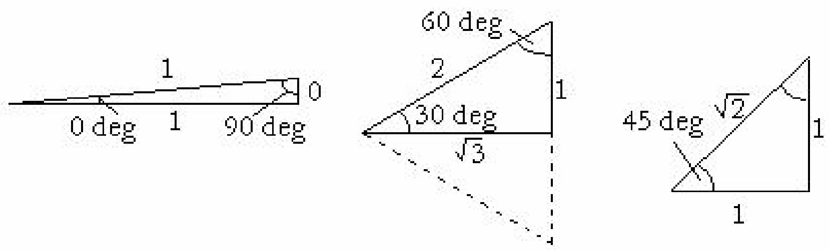


Figure 4: 0, 30, 45, 60 and 90 degree angles

$\sin 0 = 0$	$\cos 0 = 1$	$\tan 0 = 0$
$\sin 30^0 = 1/2$	$\cos 30^0 = \sqrt{3}/2$	$\tan 30^0 = 1/\sqrt{3}$
$\sin 45^0 = 1/\sqrt{2}$	$\cos 45^0 = 1/\sqrt{2}$	$\tan 45^0 = 1$
$\sin 60^0 = \sqrt{3}/2$	$\cos 60^0 = 1/2$	$\tan 60^0 = \sqrt{3}$
$\sin 90^0 = 1$	$\cos 90^0 = 0$	$\tan 90^0 = \infty$

$\mathrm{Sin}(A + B) = \mathrm{SinA\ Cos\ B} + \mathrm{CosA\ SinB}$
$\mathrm{Cos}(A + B) = \mathrm{CosA\ CosB} - \mathrm{SinA\ SinB}$
$\mathrm{Tan}(A + B) = (\mathrm{TanA} + \mathrm{TanB})/(1 - \mathrm{TanA\ TanB})$
$\mathrm{Sin}(2A) = 2\ \mathrm{SinA\ CosA}$
$\mathrm{Cos}(2A) = \mathrm{Cos}^2A - \mathrm{Sin}^2A$
$\mathrm{Tan}(2A) = 2\mathrm{TanA}/(1 - \mathrm{Tan}^2A)$
$\mathrm{SinA} + \mathrm{SinB} = 2\ \mathrm{Sin}[(½)(A + B)]\ \mathrm{Cos}[(½)(A - B)]$
$\mathrm{SinA\ CosB} = (½)[\mathrm{Sin}(A + B) + \mathrm{Sin}(A - B)]$
$\mathrm{CosA\ CosB} = (½)[\mathrm{Cos}(A + B) + \mathrm{Cos}(A - B)]$
$\mathrm{SinA\ SinB} = (½)[\mathrm{Cos}(A - B) - \mathrm{Cos}(A + B)]$
$\mathrm{SinA} - \mathrm{SinB} = 2\mathrm{Cos}\ [(½)(A + B)]\ \mathrm{Sin}\ [(½)(A - B)]$
$\mathrm{CosA} + \mathrm{CosB} = 2\mathrm{Cos}[(½)A + B)]\ \mathrm{Cos}[(½)(A - B)]$
$\mathrm{CosA} - \mathrm{CosB} = -2\mathrm{Sin}[(½)A + B)]\ \mathrm{Sin}[(½)(A - B)]$
$\mathrm{Sin}^2A - \mathrm{Sin}^2B = \mathrm{Sin}(A + B)\ \mathrm{Sin}(A - B)$
$\mathrm{Cos}^2A - \mathrm{Cos}^2B = -\mathrm{Sin}(A + B)\ \mathrm{Sin}(A - B)$
$\mathrm{Cos}^2A - \mathrm{Sin}^2B = \mathrm{Cos}(A + B)\ \mathrm{Cos}(A - B)$

Co-ordinate Geometry

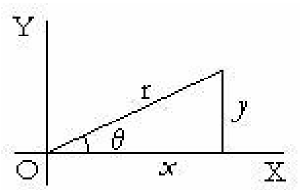


Figure 5: Cartesian and Polar Co=ordinates

Cartesian co-ordinates, points are shown by x and y
Polar co-ordinates, points are shown by r and θ
$r = \sqrt{(x^2 + y^2)}$ and $\mathrm{Tan}\theta = y/x$ and $x = r\ \mathrm{Cos}\theta$ and $y = r\ \mathrm{Sin}\theta$

Straight line, slope m $y = mx + c$
Line through (x_1, y_1) and (x_2, y_2) $y_1 = m\,x_1 + c$ and $y_2 = m\,x_2 + c$
Solve for m and c to get $(y - y_1)/(y_1 - y_2) = (x - x_1)/(x_1 - x_2)$
Angle between two lines $\mathrm{Tan}\theta = (m_1 - m_2)/(1 + m_1 m_2)$
2 lines cross orthogonally if $m_1\, m_2 = -1$
Circle, centre at origin $x^2 + y^2 = a^2$
Circle, centre at (g,h), radius a $(x - g)^2 + (y - h)^2 = a^2$
Ellipse, centre at origin $x^2/a^2 + y^2/b^2 = 1$
Parabola $y^2 = 4ax$
Hyperbola $xy = c^2$ or $x^2/a^2 - y^2/b^2 = 1$

Logarithms

By definition of a Log $\log_a m = x$ where $a^x = m$
Hence $\log_a m + \log_a n = \log_a(mn)$
$\log_a m - \log_a n = \log_a(m/n)$
$n \log_a m = \log_a m^n$
$\log_b m = \log_a m \,/\, \log_a b$

Binominal

$(x + a)^n$
$= x^n + n\,a\,x^{n-1} + [n\,(n-1)/2!]\,a^2\,x^{n-2} + [n\,(n-1)\,(n-2)/3!]\,a^3\,x^{n-3} \ldots + n!/[(n-r)!\,r!]\,a^r\,x^{n-r} \ldots + a^n$
Put $x = 1$ and $a = x$
$(1 + x)^n = 1 + nx + [n\,(n-1)/2!]\,x^2 + [n\,(n-1)\,(n-2)/3!]\,x^3 \ldots + n!/[(n-r)!\,r!]\,x^r + \ldots x^n$
This is found to be valid with negative or fractional values for n provided $1 > x > -1$

Matrices

Data can be displayed and manipulated in short hand in the form of Matrices.

$a_1 x + a_2 y + a_3 z + a_4 = 0$ can be written in Matrix form $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix} = 0$

$b_1 x + b_2 y + b_3 z + b_4 = 0$

$c_1 x + c_2 y + c_3 z + c_4 = 0$

(add or subtract lines to get coefficients 2 and 3 = 0 to solve for x etc)

Determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

terms in sequence abc and numbers in sequence 123123 positive, others negative

Series

Definitions;

Arithmetical Progression AP is a Series with the same difference between all adjacent terms

Geometrical Progression GP is a Series with the same ratio between all adjacent terms

Sum S of AP, 1st term a, difference d, n terms

Add first term to last term, 2nd term to 2nd last etc hence $2S = n[\{a\} + \{a + (n - 1)d\}]$

Sum S of GP, 1st term a, ratio of terms p, n terms.

Then Series - p times Series = first term + last term hence $S = a(1 - p^n)/(1 - p)$

Sum of first n numbers is an AP = $n(n + 1)/2$

Sum of first n squares = $(1/6)n(n + 1)(2n + 1)$

Sum of first n cubes = $[(n + 1)n/2]^2$

Calculus

Definitions;

The Differential of $[y = f(x)]$ written dy/dx is the Slope of $f(x) = 0$.

The Integral of y (written $\int y\,dx$) is the sum of areas of height y and width dx

$d/dx[ax^n] = a\,n\,x^{n-1}$

$\int a x^n\,dx = a x^{n+1}/(n+1) + c$

Integration between limits is the value between two specified values of x.

In polar co-ordinates, $dy/dx = (\sin\theta\,dr/d\theta + r\cos\theta)/(\cos\theta\,dr/d\theta - r\sin\theta)$

In polar co-ordinates, Sum of areas = $\int(\tfrac{1}{2})\,r^2\,d\theta$

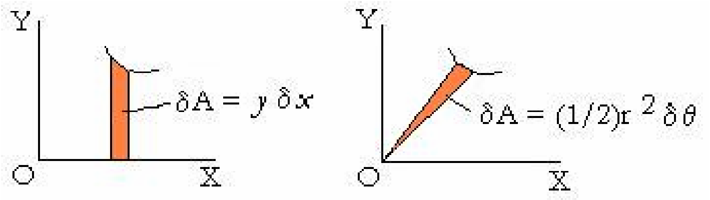


Figure 6: Elemental Areas

Differential of a sum $d/dx\ (u + v) = du / dx + dv / dx$
Differential of a product $d/dx\ (u\ v) = v\ du/dx + u\ dv/dx$
Differential of a Fraction (put $v = v^{-1}$) $d/dx\ (u / v) = \{ v\ du / dx - u\ dv / dx\} / v^2$
Differential with a change of variable $dy / dx = dy / du\ .\ du / dx$

ln is natural logarithm (ie Log to base e) where $e = 1 + 1/1! + 1/2! + 1/3! + \ldots$ to infinity
By definition, if $\ln(m) = x$ then $e^x = m$
$e^x = 1 + x/1! + x^2/2! + x^2/3! + \ldots$ to infinity
$d/dx\ (e^x) = e^x$ and $\int (1/x)\ dx = \ln(x) + c$ and $d/dx\ [\ln(x)] = 1/x$
$a^x = e^{x \ln(a)}$
$d/dx\ (\mathrm{Sin}x) = \mathrm{Cos}x$ and $d/dx\ (\mathrm{Cos}x) = -\mathrm{Sin}x$ and $d/dx\ (\mathrm{Tan}x) = \mathrm{Sec}^2 x$
$d/dx\ \{\mathrm{Arc\ Sin}(x/a)\} = 1/\sqrt{(a^2 - x^2)}$ and $d/dx\ \{\mathrm{Arc\ Cos}(x/a)\} = -1/\sqrt{(a^2 - x^2)}$
$d/dx\ \{\mathrm{Arc\ Tan}(x/a)\} = a/(a^2 + x^2)$

MacLaurim's Theorem

Let $f(x) = a_0 + a_1 x/1! + a_2 x^2/2! + \ldots + a_r x^r/r! +$
Write $f_r(0)$ to mean r th differential of $f(x)$ with x then made zero, hence $a_r = f_r(0)$
$f(x) = f(0) + f_1(0)\ x/1! + f_2(0)\ x^2/2! + f_3(0)\ x^3/3! + \ldots + f_r(0)\ x^r/r! +$
The series for many functions can be written down, eg

$\mathrm{Sin}\ x = x - x^3/3! + x^5/5! - x^7/7! \ldots\ldots$
$\mathrm{Cos}\ x = 1 - x^2/2! + x^4/4! - x^6/6! \ldots\ldots$
$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! \ldots\ldots$
$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 \ldots\ldots$

Taylor's Theorem

$f(x) = f(a) + (x - a) f_1(a) + \ldots + [(x - a)^r / r!\ ] f_r(a) + \ldots$

Hyperbolic Functions

Expand $\mathrm{Cos}(n\theta) + i\ \mathrm{Sin}(n\theta)$ by MacLaurim's Theorem and the result is the expansion of $e^{in\theta}$
$\mathrm{Cos}(n\theta) + i\ \mathrm{Sin}(n\theta) = e^{in\theta} = [\mathrm{Cos}\theta + i\ \mathrm{Sin}\theta]^n$
$\mathrm{Cos}\theta + i\ \mathrm{Sin}\theta = e^{i\theta}$ and $\mathrm{Cos}\theta - i\ \mathrm{Sin}\theta = e^{-i\theta}$
$\mathrm{Cos}\theta = \{e^{i\theta} + e^{-i\theta}\} / 2$ and $\mathrm{Sin}\theta = \{e^{i\theta} - e^{-i\theta}\} / 2i$

By definition, Cosh and Sinh are these values of Cos and Sin without the complex number i

$\mathrm{Cosh}\theta = \{e^{\theta} + e^{-\theta}\} / 2$ and $\mathrm{Sinh}\theta = \{e^{\theta} - e^{-\theta}\} / 2$

$\mathrm{Tanh}\theta = (\mathrm{Sinh}\theta) / (\mathrm{Cosh}\theta)$ and $\mathrm{Sech}\theta = 1/\mathrm{Cosh}\theta$

$\mathrm{Cosech}\theta = 1/\mathrm{Sinh}\theta)$ and $\mathrm{Coth}\theta = 1/\mathrm{Tanh}\theta$

$\mathrm{Cosh}^2\theta - \mathrm{Sinh}^2\theta = 1$ and $1 - \mathrm{Tanh}^2\theta = \mathrm{Sech}^2\theta$

$\mathrm{Sinh}(2\theta) = 2\,\mathrm{Sinh}\theta\ \mathrm{Cosh}\theta$ and $\mathrm{Cosh}(2\theta) = \mathrm{Cosh}^2\theta + \mathrm{Sinh}^2\theta$

$d/d\theta\ (\mathrm{Sinh}\theta) = \mathrm{Cosh}\theta$ and $d/d\theta\ (\mathrm{Cosh}\theta) = \mathrm{Sinh}\theta$

$d/d\theta\ (\mathrm{Tanh}\theta) = \mathrm{Sech}^2\theta$

Methods for Integration

In General Look for a substitution that will simplify the integral
$\int F(ax \pm b)dx$ indicates the substitution $u = (ax \pm b)$ thus $du = a\,dx$
eg $\int[\mathrm{Sin}(x + a)]\,dx$ Put $u = x + a$ thus Integral $= \int[\mathrm{Sin}(u)]du = -\mathrm{Cos}(u) +$ constant
$\int[1/(x^2 + a^2)]dx$ indicates the substitution $x = a\,\mathrm{Tan}(u)$ or $x = a\,\mathrm{Sinh}(u)$
eg $\int[1/(x^2 + a^2)]dx$ Put $x = a\,\mathrm{Tan}(u)$ this leads to $(1/a)\int du = u/a +$ constant

Fractions If the denominator factorizes, Split into Partial Fractions;
$\int[1/\{(x \pm a)(x \pm b)\}]dx = \int[A/(x \pm a)]dx + \int[B/(x \pm b)]dx$
eg $\int[1/(x^2 - a^2)]dx = \int[(1/2a)/(x - a)]dx - \int[(1/2a)/(x + a)]dx$
$= (1/2a)[\ln(x - a) - \ln(x + a)] + C$

Integrals of Square Roots
Remember $1 - \mathrm{Sin}^2u = \mathrm{Cos}^2u$, $1 + \mathrm{Tan}^2u = \mathrm{Sec}^2u$, $1 + \mathrm{Sinh}^2u = \mathrm{Cosh}^2u$, and $\mathrm{Cosh}^2u - 1 = \mathrm{Sinh}^2u$
$\int[1/\sqrt{(a^2 - x^2)}]dx$ indicates the substitution $x = a\,\mathrm{Sin}\,(u)$ therefore $dx = a\,\mathrm{Cos}\,(u)\,du$
$\int[1/\sqrt{(a^2 - x^2)}]dx = \int[1/a\mathrm{Cos}(u)]a\mathrm{Cos}(u)du = \int du = u +$ constant
$\int[1/\sqrt{(a^2 + x^2)}]dx$ indicates the substitution $x = a\,\mathrm{Sinh}(u)$ or $x = a\,\mathrm{Tan}(u)$
$\int[1/\sqrt{(x^2 + a^2)}]dx$ Put $x = a\,\mathrm{Sinh}(u)$ this leads to $\int du = u +$ constant
$\int[1/\{\sqrt{(x^2 - a^2)}\}]dx$ indicates the substitution $x = a\,\mathrm{Cosh}\,(u)$
$\int[1/\{\sqrt{(x^2 - a^2)}\}]dx$ Put $x = a\,\mathrm{Cosh}\,u$ leads to $\int du = u +$ constant
$\int[1/\{\sqrt{(ax^2 + bx + c)}\}]dx$ Remove the x term,
Put $a[(x + p)^2 + q] = ax^2 + bx + c$ Equate coefficients to solve for p and q,
Put $u = x + p$ and $r^2 = q$.
This leads to $(1//a)\int[1 / \sqrt{(u^2 \pm r^2)}]du$ As above put $u = r\,\mathrm{Sinh}\,v$ or $u = r\,\mathrm{Cosh}\,v$

Trigonometrical integrals
(i)Put in form $\int F(u)\,du$
for example $\int F(\mathrm{Cos}\,x)\,\mathrm{Sin}\,x\,dx$, or $\int F(\mathrm{Sin}\,x)\,\mathrm{Cos}\,x\,dx$ or $\int F(\mathrm{Tan}\,x)\,\mathrm{Sec}^2 x\,dx$
Similarly for hyperbolics
for example $\int \mathrm{Sinh}^3 x\,dx = \int(\mathrm{Cosh}^2 x - 1)\,\mathrm{Sinh}\,x\,dx = 1/3\,\mathrm{Cosh}^3 x - \mathrm{Cosh}\,x +$ constant
or (ii) Try $u = \mathrm{Tan}(x)$ since $dx = du/(1 + u^2)$
or (iii) Try $t = \mathrm{Tan}(x/2)$ since $dx = 2\,dt/(1 + t^2)$, $\sin(x) = 2t/(1 + t^2)$
and $\mathrm{Cos}(x) = (1 - t^2)/(1 + t^2)$. All have the same Denominator which may cancel.
$\int[1/(a\,\mathrm{Sin}\,x + b\,\mathrm{Cos}\,x + c)]dx$ indicates the substitution $t = \mathrm{Tan}(x/2)$

$\int \mathrm{Cos}^2(x)\,dx$ and $\int \mathrm{Sin}^2(x)\,dx$ indicate $u = 2x$ since $\mathrm{Cos}^2(x) = ½\,[\mathrm{Cos}(u) + 1]$ and $dx = (½)\,du$

1/D Method

The operator D is defined as d/dx.

$D(y) = dy/dx$ hence $(D + a)(D + b)(y) = D^2(y) + (a + b)D(y) + ab\,y$

$D^{-1}(y) = \int y\,dx$

$D^n (e^{ax} V) = e^{ax} (D + a)^n V$

$[1/F(D)]\, e^{ax} = [1/F(a)]e^{ax}$

$F(D^2)\ (a \operatorname{Sin} mx + b \operatorname{Cos} mx) = F(-m^2)\ (a \operatorname{Sin} mx + b \operatorname{Cos} mx)$

$\int e^{ax}\operatorname{Cos}(bx)\,dx$ and $\int e^{ax}\operatorname{Sin}(bx)$ can be integrated by the 1/D method

but it is simpler to consider the Real (or Complex) part of $\int e^{ax}[\operatorname{Cos}(bx) + i \operatorname{Sin}(bx)]\,dx$
$= \int e^{(a+ib)x}dx = [1/(a + ib)]e^{(a+ib)x}$ + constant

Integration by Parts

$d/dx\,(u\,v) = v\,du/dx + u\,dv/dx$, therefore $\int u\,dv = uv - \int v\,du$

Use to transform the Integral of a product

example (i) $\int x \operatorname{Sin}(x)\,dx$ Put $x = u$ and $\operatorname{Sin}(x)\,dx = dv$ therefore $v = -\operatorname{Cos}(x)$ and $du = dx$

example(ii) $\int x \ln(x)\,dx$ Put $\ln(x) = u$ and $x\,dx = dv$ therefore $v = (½)\,x^2$ and $du = 1/x\,dx$

Table 1: Standard Forms

y	dy / dx	$\int y\,dx$
$a\,x^n$	$n\,a\,x^{n-1}$	$a\,x^{n+1}/(n + 1)$
a / x	$-a / x^2$	$a \ln x$
$\operatorname{Sin}(\omega x)$	$\omega\operatorname{Cos}(\omega x)$	$(-1/\omega) \operatorname{Cos}(\omega x)$
$\operatorname{Cos}(\omega x)$	$-\omega\operatorname{Sin}(\omega x)$	$(1/\omega) \operatorname{Sin}(\omega x)$
$\operatorname{Tan}(\omega x)$	$\omega\operatorname{Sec}^2(\omega x)$	$-(1/\omega) \ln \{\operatorname{Cos}(\omega x)\}$
$\operatorname{Sec} x$	$\tan x \operatorname{Sec} x$	$\ln(\operatorname{Sec} x + \operatorname{Tan} x)$
Cosec x	– Cot x Cosec x	ln (Cosec x – Cot x)
$\operatorname{Cot} x$	$-\operatorname{Cosec}^2 x$	$\ln(\operatorname{Sin} x)$
$\operatorname{Arc Sin}(x/a)$	$1/\sqrt{(a^2 - x^2)}$	$x \operatorname{Arc Sin}(x/a) + \sqrt{(a^2 - x^2)}$
$\operatorname{Arc Cos}(x/a)$	$-1/\sqrt{(a^2 - x^2)}$	$x \operatorname{Arc Cos}(x/a) - \sqrt{(a^2 - x^2)}$
$\operatorname{Arc Tan}(x/a)$	$a/(a^2 + x^2)$	$x \operatorname{Arc Tan}(x/a) - a \ln\sqrt{(a^2 + x^2)}$
e^{ax}	$a\,e^{ax}$	$(1/a)\,e^{ax}$
a^x	$a^x \ln a$	$a^x/(\ln a)$
$\ln(a\,x)$	$1 / x$	$x\,[\ln(a\,x) - 1]$
$\operatorname{Log}_a x$	$(1/x) \operatorname{Log}_a e$	$x \operatorname{Log}_a (x/e)$
$\operatorname{Sinh} x$	$\operatorname{Cosh} x$	$\operatorname{Cosh} x$
$\operatorname{Cosh} x$	$\operatorname{Sinh} x$	$\operatorname{Sinh} x$
$\operatorname{Tanh} x$	$\operatorname{Sech}^2 x$	$\ln(\operatorname{Cosh} x)$
$\operatorname{Arc Sinh}(x/a)$	$1/\sqrt{(a^2 + x^2)}$	$x \operatorname{Arc Sinh}(x/a) - \sqrt{(a^2 + x^2)}$
$\operatorname{Arc Cosh}(x/a)$	$1/\sqrt{(x^2 - a^2)}$	$x \operatorname{Arc Cosh}(x/a) - \sqrt{(x^2 - a^2)}$
$\operatorname{Arc Tanh}(x/a)$	$a/(a^2 - x^2)$	$x \operatorname{Arc Tanh}(x/a) + a \ln\sqrt{(a^2 - x^2)}$

Functions of Time and other variables

Velocity $v = dx/dt$ and Acceleration $= d^2x/dt^2 = v\,dv/dx$
Speed of rotation $d\theta/dt = \omega$ and Angular acceleration $d^2\theta/dt^2 = d\omega/dt$

Functions of two or more variables

$V = F(x,y,z)$ therefore $\delta V = (\partial V/\partial x)\,\delta x + (\partial V/\partial y)\,\delta y + (\partial V/\partial z)\,\delta z$
where $\partial V/\partial x$ means the differential of V with respect to x while y and z are kept constant.

Areas and Volumes

Surface Area of sphere = $4\pi R^2$ = Curved area of enclosing cylinder
Volume of cone and pyramid = (1/3) (Base Area) x (Height)
Volume of cylinder = $\pi R^2 h$
Volume of sphere = $(4/3)\pi R^3$
Volume of Revolution, ie volume enclosed by rotating a curve about the x axis

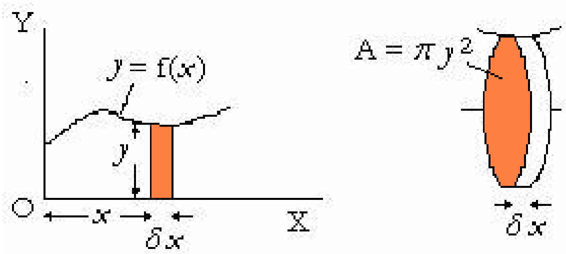


Figure 7: Volume of Revolution
Volume of Revolution= $\int \pi y^2 \, dx$

Maxima and Minima

$y = F(x)$ is a Maximum when $dy/dx = 0$ and d^2y/dx^2 is negative
$y = F(x)$ is a Minimum when $dy/dx = 0$ and d^2y/dx^2 is positive
$y = F(x)$ is a point of inflection when $dy/dx = 0$ and $d^2y/dx^2 = 0$
$y = F(x,y)$ is a Maximum when $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ and $\partial^2 F/\partial x^2$ is negative
and $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\,\partial y]^2$
$y = F(x,y)$ is a Minimum when $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ and $\partial^2 F/\partial x^2$ is positive
and $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\,\partial y]^2$

Graphs

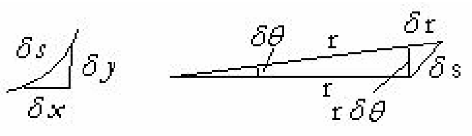


Figure 8: Length of arc

Length of Arc $s = \int\sqrt{[1 + (dy/dx)^2]}dx = \int\sqrt{[r^2 + (dr/d\theta)^2]}d\theta$

Radius of Curvature $\rho = [1 + (dy/dx)^2]^{3/2} / (d^2y/dx^2)$

Vectors

Definitions;

Scalar has Magnitude but not Direction. Vector has magnitude and Direction

The Operator j rotates a vector 90^0 anticlockwise, $j^2\,\mathbf{V} = -\mathbf{V}$

therefore $j = \sqrt{(-1)} = i$, is one solution

The Operator h rotates a vector 120^0 anticlockwise, hence $h^3\,\mathbf{V} = \mathbf{V}$ and $(1 + h + h^2)\,\mathbf{V} = 0$

i, **j** and **k** are three vectors mutually at right angles each length one unit.

Shake Hands, Right Hand, Fingers point as **i** , Palm points as **j** , Thumb points as **k** (Go tip to thumb)

Matrix notation of a Vector. $|\ a_i\ \ a_j\ \ a_k\ |$ means Vector $a_i\,\mathbf{i} + a_j\,\mathbf{j} + a_k\,\mathbf{k}$

If $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ then $V = \sqrt{[V_x^{\,2} + V_y^{\,2} + V_z^{\,2}]}$

Let θ be the angle between two vectors $\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$ and $\mathbf{U} = U_x\,\mathbf{i} + U_y\,\mathbf{j} + U_z\,\mathbf{k}$

By definition, $\mathbf{V} \bullet \mathbf{U} = V\,U\,\mathrm{Cos}\,\theta$ where θ is the angle between **V** and **U**

Therefore $\mathbf{V} \bullet \mathbf{U} = V_xU_x + V_yU_y + V_zU_z$ and $\mathbf{V} \bullet \mathbf{U}$ is a Scalar

$\mathrm{Cos}\,\theta = [\,V_xU_x + V_yU_y + V_zU_z\,]/\sqrt{[\{V_x^{\,2} + V_y^{\,2} + V_z^{\,2}\}\{U_x^{\,2} + U_y^{\,2} + U_z^{\,2}\}]}$

V and **U** are orthogonal if $V_xU_x + V_yU_y + V_zU_z = 0$

$\mathrm{Cos}^2\,\alpha + \mathrm{Cos}^2\,\beta + \mathrm{Cos}^2\,\gamma = 1$ where β ,α and γ are the angles between a vector and each axis.

By definition, $\mathbf{V}\ \mathbf{X}\ \mathbf{U} = V\,U\,\mathrm{Sin}\,\theta\,\mathbf{a}$ where θ is the angle between **V** and **U** and

a is a unit vector orthogonal to **V** and **U** hence $\mathbf{V}\ \mathbf{X}\ \mathbf{U}$ is a Vector

$\mathbf{V}\ \mathbf{X}\ \mathbf{U}$ = the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ V_x & V_y & V_z \\ U_x & U_y & U_z \end{vmatrix}$$

If **A**, **B** and **C** define three adjacent edges of a parallelepiped
then Volume = $\mathbf{A}\ \mathbf{XB} \bullet \mathbf{C} = \mathbf{B}\ \mathbf{XC} \bullet \mathbf{A} = \mathbf{C}\ \mathbf{XA} \bullet \mathbf{B}$

If a scalar value F is assigned to all points in a three dimensional volume, then by definition, **Grad F** (written **ΔF**) at any point is a Vector normal to the surface which connects the point to adjacent points which have the same value of F. **ΔF** has the magnitude equal to the differential of F with respect to distance in this direction

If a Vector **F** is assigned to all points in a 3D volume, then its differential is a Vector.
Div **F** is defined as $\partial \mathbf{F}/\partial \bullet \mathbf{i} + \partial \mathbf{F}/\partial y \bullet \mathbf{j} + \partial \mathbf{F}/\partial z \bullet \mathbf{k} = \Delta \bullet \mathbf{F}$ and is a scalar.
and Curl **F** is defined as $\partial \mathbf{F}/\partial x\ \mathbf{Xi} + \partial \mathbf{F}/\partial y\ \mathbf{Xj} + \partial \mathbf{F}/\partial z\ \mathbf{Xk} = \mathbf{X}\ \Delta\mathbf{F}$ and is a vector.

Argand Diagram

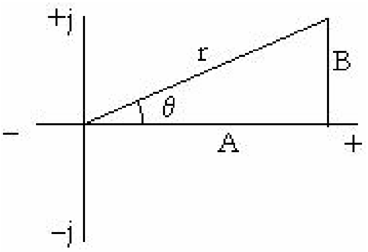


Figure 9: Argand Diagram

The Complex Number A + iB can be represented as a Vector A + jB

$A + iB = r[\cos\theta + i\sin\theta] = r\,e^{i\theta}$

where $r = \sqrt{(A^2 + B^2)}$ and $\theta = \text{Arc Tan}(B/A)$

Thus $[A + iB]^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)]$

Use for Multiplication or Division by Complex Numbers (and Vectors)

$\cos(2\pi n + \theta) + i\sin(2\pi n + \theta) = \cos\theta + i\sin\theta$ where n is any integer

Do not confuse the operators i or j in an Argand diagram with the unit vectors **i** and **j**.

(Operator $j)^2 = -1$ but (unit vector $\mathbf{j})^2 = 1$

Differential Equations

Definitions;

Ordinary or Partial (2 or more variables)

Order, if highest derivative is d^ny/dx^n Order is n

Arbitrary Constants. Solution has as many arbitrary constants as the Order.

Constants can be evaluated by initial or final conditions.

Degree is the Index of the highest derivative when rationalised

PI is the Particular Integral

CF is the Complementary Function

Complete Primitive = PI + CF

Singular Solution is an isolated solution

Linear Differential Equation. Each term a Differential of y, all Degree one, Coefficients are functions of x

(i) Solution of a Linear Differential Equation

Put $y = a_0 + a_1 x + a_2 x^2/2! + a_3 x^3/3! + \ldots + a_r x^r/r! + \ldots$

Check the answer has enough arbitrary constants

(ii) Exact Equations (first order) $M\,dx + N\,dy = 0$ can be integrated immediately if $\partial N/\partial x = \partial M/\partial y$

(iii) Separate the variables to get $P(x)\,dx = Q(y)\,dy$

for example. $f(x)\,dy/dx = a$ then $y = \int[a/f(x)]dx + c$

(iv) Homogeneous Equations $dy/dx = f(y/x)$ Put $y = vx$

(v) Linear first order $dy/dx + P(x)\,y = Q(x)$ where $P(x)$ and $Q(x)$ are any function of x
multiply by integrating factor $R = e^{\int P\,dx}$

(vi) Linear, constant coefficients $F(D)y = f(x)$ for example $7D^2(y) - 3D(y) + 9y = 2 + 3x$
CF Solve $F(D)y = 0$. Put $y = A\,e^{ax} + Be^{bx} +$ etc where A, B, etc are arbitrary constants
Special cases (a) a and b conjugate pair $p \pm iq$, $y = e^{px}\,[A \text{ Cos } (qx) + B \text{ Sin } (qx)]$
(b) $a = b$, $y = A\,e^{ax} + Bxe^{bx}$
PI Find one solution to $F(D) = f(x)$ and add to the CF to get the complete solution
Examples to find a PI
(a) $f(x) = k_0 + k_1x + k_2x^2 +$ etc Put $y = a_0 + a_1x + a_2x^2$ etc and equate coefficients of x
(b) $f(x) = k \sin x$ or $k \cos x$ Put $y = a_1 \sin x + a_2 \cos x$
or take real (or complex) part of $y = a\,e^{ibx}$

(vii) $d^2y/dx^2 = -A\,y$ This is SHM. Solve by multiplying by the integrating factor $2\,dy/dx$

(viii) Solution by Laplace Transform solves for f(t) and evaluates the arbitrary constants
Used for evaluating the response of a control system
If $f(t) = A\,t^n\,e^{-at}$ then Laplace Transform $F(s) = A\,n!/(a + s)^{n+1}$
If $f(t) = A\,t^n \text{ Sin } \omega t$ or $A\,t^n \text{ Cos } \omega t$ then $F(s) =$ Real or Complex part of $A\,n!/(s - j\omega)^{n+1}$
Laplace Transform of $d/dt\,[f(t)] = s\,F(s) - f(0)$
Laplace Transform of $d^2/dt^2\,[f(t)] = s^2\,F(s) - sf(0) - d/dt[f(0)]$
Laplace Transform of $\int f(t)\,dt = (1/s)\,F(s)$

(ix) Bessell's Eqation. $x^2\,d^2y/dx^2 + x\,dy/dx + (x^2 - n^2)\,y = 0$
where $n = 0.\ 1,\ 2,\ 3,\ 4, \ldots$ etc or $n = 1/2,\ 1/3,\ 1/4, \ldots$ etc
The Solution is $y = A\,J_n(x) + B\,Y_n(x)$ where A and B are arbitrary constants
$J_n(x) = \sum[\ \{(-1)^s\,(x/2)^{2s+n}\}\ /\ \{\ \Gamma(s + n + 1)\ s!\ \}\ ]$ from $s = 0$ to infinity
For positive integers $\Gamma(x) = (x - 1)!$ for other values, $\Gamma(x) = \int t^{(x-1)}\,e^{-t}\,dt$ from $t = 0$ to ∞
$Y_n(x) = [\,\text{Cos } n\pi\ J_n(x) - J_{-n}(x)\,]\ /\ \text{Sin } n\pi$
$d^2y/dx^2 + x\,y = 0$ can be converted to Bessell's Eqtn by substitution

Fourier Series

Any cyclic function $y = F(x)$ can be converted to a series of the form
$y = c_0 + a_1 \text{ Cos } x + a_2 \text{ Cos } 2x + \ldots\ a_n \text{ Cos } nx + \ldots + b_1 \text{ Sin } x + b_2 \text{ Sin } 2x + \ldots\ b \text{ Sin } nx + \ldots$

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y\,\text{Cos}(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y\,\text{Sin}(nx)\,dx$$

SUMMARY (PART 2 APPLIED)

Mechanics

Constant acceleration equations

$v = u + at$ $\quad$ $s = (½)(u + v)\,t$ $\quad$ $s = ut + (½)\,at^2$ $\quad$ $v^2 = u^2 + 2as$

Gravitational Force $\quad F = G\,M_1 M_2 / d^2$

Moment of Inertia $\quad I = \int x^2\,dm$

Newton's Laws (summarised)

(i) A body moves in a straight line unless acted on by a force

(ii) $P = ma$ and $C = I\,d\omega/dt = I\,d^2\theta/dt^2$

(iii) Action and Reaction are equal and opposite

Conservation of Energy

Work done $= F\,x = C\,\theta$

Kinetic Energy $= (½)\,m\,v^2 = (½)\,I\,\omega^2$

Potential Energy $= m\,g\,h$

Conservation of Momentum Momentum $= m\,v$ and Angular momentum $= I\,\omega$
(u before, v after collision) $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$ and $v_1 - v_2 = -e\,(u_1 - u_2)$
Body moving in a curve acceleration $= v^2/R = R\,\omega^2$
Two Dimension Forces in equilibrium Resultant Force in two directions = 0
Plus Couple about any one point = 0
Three Forces in equilibrium are co-planar and either meet at a point or are parallel
Friction Force = μN under gravity $F = \mu mg$

Simple Harmonic Motion (SHM) $d^2x/dt^2 = -k\,x$
Therefore $x = a \sin \omega t + b \cos \omega t$ where $\omega = \sqrt{k}$
T the Time for one cycle (ie the Period) is given by $\omega T = 2\pi$, therefore Period $= 2\pi/\sqrt{k}$

Capstan $P_2 = P_1 e^{\mu\theta}$

Structures Stress = p/A Strain = x/L E = Stress/Strain

Beam carrying a load $p/y = E/R = M/I$
where
p is stress at distance y from the Neutral Axis
E is Young's Modulus
R is radius of curvature
M is the bending moment
I is the 2nd moment of area about the Neutral Axis

Cantilever Beam

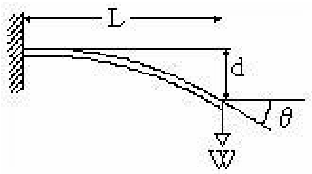


Figure 10: Cantilever Beam

Moment M at L $d = ML^2/(2EI)$ $\theta = ML/(EI)$
Load W at L $d = WL^3/(3EI)$ $\theta = WL^2/(2EI)$
Distributed Load W $d = WL^3/(8EI)$ $\theta = WL^2/(6EI)$

Suspension Bridge Parabolic $y = w\,x^2/(2F)$
Hanging chain Catenary $y = c\,[\cosh(x/c) - 1]$

Gyroscopes **C = ω X M**

Where **C** is a couple expressed as a corkscrew vector
ω is the angular velocity of precession expressed as a corkscrew vector
M is the angular momentum of the flywheel expressed as a corkscrew vector

Longitudinal and Hoop Stress

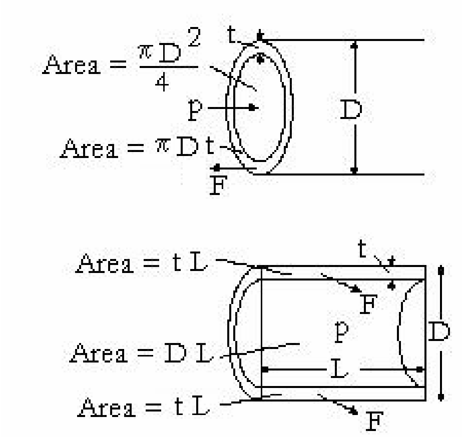


Figure 11: Longitudinal and Hoop Stress

Longitudinal stress = $(p\pi D^2/4) / \pi Dt = pD/4t$

Hoop stress = $pDL/2tL = pD/2t$

where p is pressure, D is outside dia, t is wall thickness
Use the outside diameter to allow for radial compression in the shell

PART 1: PURE MATHEMATICS

1 ARITHMETIC

Terminology

The **Sum** of a set of numbers is the addition of all the numbers.
The **Difference** between two numbers is one minus the other.
The **Product** of a set of numbers is one of the numbers times all the others
The **Quotient** is the answer when one number is divided by another

Mathematical Symbols

The symbol + is used for "plus" (ie add)
The symbol − is used for "minus" (ie subtract)
The symbol x or a blank space is used in this book for "multiplied by". Some textbooks use a dot.
Computer languages use the asterisk * to prevent confusion with the letter x.
The symbol / is used for "divided by" (the traditional symbol is ÷)
Brackets are used to show the order in which an expression is evaluated. The expression inside the brackets is evaluated first.
If brackets are not shown, x and / are evaluated before + and − .
Thus 3 + 4 x 5 = 3 + 20, not 7 x 5
Nested brackets can be [{ ( ) }], but computers use ((( ))).

Long Multiplication

Multiplication is done by calculator but it can be done manually one digit at a time, called Long Multiplication, as follows;

Example 12345×6789

$$
\begin{array}{rl}
1\,2\,3\,4\,5 & \\
\underline{6\,7\,8\,9} & \\
7\,4\,0\,7\,0 & = 6 \times 12345 \\
8\,6\,4\,1\,5 & = 7 \times 12345 \\
9\,8\,7\,6\,0 & = 8 \times 12345 \\
\underline{1\,1\,1\,1\,0\,5} & = 9 \times 12345 \\
8\,3\,8\,1\,0\,2\,0\,5 & = \text{sum of numbers above}
\end{array}
$$

Multiplying by a negative number

Doubling a negative number gives a negative number twice the size

Therefore (positive number) $\times$ (negative number) $=$ (negative number)

Multiplying by a negative number changes the sign of the other number.

Hence $5 \times (-2) = -10$

and $(-5) \times (-2) = +10$

Long Division

Division is done by calculator but it can also be done manually, called Long Division. This is best shown by an example. $12345 \div 678$

$$
\begin{array}{r|rl}
 & \overline{1\,8} & \\
6\,7\,8 & 1\,2\,3\,4\,5 & \\
 & \underline{6\,7\,8} & = 1 \times 678 \quad \text{enter 1 on the top line} \\
 & 5\,5\,6\,5 & = 1234 - 678 \text{ plus 5 brought down} \\
 & \underline{5\,4\,2\,4} & = 8 \times 678 \quad \text{enter 8 on the top line} \\
 & 1\,4\,1 & = 5565 - 5424
\end{array}
$$

Answer $12345 \div 678 = 18$ Remainder 141

Factors

If a number can be divided by another with no remainder, the second number is a factor of the first.

Example $12 = 2 \times 2 \times 3$. Therefore 2, 3, 4 and 6 are factors of 12

Prime Numbers

Prime Numbers

Prime numbers are numbers that have no factors except 1 and itself.

A number is divisible by 2 if it is an even number (ie ends with 0, 2, 4, 6, or 8)
A number is divisible by 3 if the sum of all the digits is divisible by 3
A number is divisible by 5 if it ends in 0 or 5
A number is divisible by 11 if the sum of alternate digits are the same or differ by a multiple of 11.

Examples

123456	The last digit is 6, therefore 2 is a factor
12339	The sum of the digits is 18, therefore 3 is a factor
693	The 1st and 3rd digits add to 9. The 2nd digit is 9. Therefore 11 is a factor
496969	$4 + 6 + 6 = 16$. $9 + 9 + 9 = 27$. Sums of alternate digits differ by 11, therefore 11 is a factor

Examples of prime numbers

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37 41 etc

Highest Common Factor (HCF)

The HCF of two (or more) numbers is the highest factor that is common to both (or all).

Example Find the HCF of 16 and 24

$16 = 2 \times 2 \times 2 \times 2$

$24 = 2 \times 2 \times 2 \times 3$

$\text{HCF} = 2 \times 2 \times 2 = 8$

Lowest Common Multiplier (LCM)

The LCM of two (or more) numbers is the lowest number that has both (or all) numbers as a factor.

Example Find the LCM of 9, 12 and 26

$9 = 3 \times 3$

$12 = 2 \times 2 \times 3$

$26 = 2 \times 13$

$\text{LCM} = 3 \times 3 \times 2 \times 2 \times 13 = 468$

Fractions and Decimals

A fraction is a value expressed as one number divided by another. A decimal is the value expressed in tenths, plus hundreds, plus thousandths etc.

Example $\frac{5}{8}$ is a fraction. Its value as a decimal is 0.625

A fraction has a **Numerator** and a **Denomiator**. (The Numerator is above the line and the Denominator is Down below)

The Reciprocal of a number is one divided by the number. To get the reciprocal of a fraction, the numerator and denominator swap places.

To multiply two fractions, the numerator of the result is the product of the two numerators and the denominator is the product of the two denominators.

To divide a fraction by another fraction, multiply by the reciprocal.

Turn the Fraction you're dividing by
Upside down and Multiply

To add or subtract fractions, multiply the numerator and denominator of each fraction by a factor to bring its denominator up to the LCM of all the denominators.

Examples

(i) $\frac{2}{3} \times \frac{1}{2} = \frac{2 \times 1}{3 \times 2} = \frac{2}{6} = \frac{1}{3}$

(ii) $\frac{2}{3} \div \frac{1}{2} = \frac{2 \times 2}{3 \times 1} = \frac{4}{3} = 1\frac{1}{3}$

(iii) $5 \times 6\frac{2}{3}$ $\quad 6\frac{2}{3} = \frac{18+2}{3} = \frac{20}{3}$

$5 \times 6\frac{2}{3} = \frac{5 \times 20}{3} = \frac{100}{3} = 33\frac{1}{3}$

(iv) $\frac{1}{3} + \frac{1}{6}$ LCM of the Denominators is 6

$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$

Recurring Decimals

The Decimal value of 1/3 = 0.333333333333 and on and on for ever
This is called 0.3 recurring

Factorials

The Factorial of a number is the product of all the numbers from 1 up to the number and is denoted by the exclamation mark.

Thus 6 ! = 1 x 2 x 3 x 4 x 5 x 6 = 720

Ratios

The ratio is the relationship of two or more numbers. The numbers are written with a colon between each

The Ratio 15 : 5 is the same as the Ratio 3 : 1 ie the first number is 3 times the second number.

Example £280 is shared between four people in the Ratio 3 : 2 : 5 : 4
The sum of the shares is 3 + 2 + 5 + 4 = 14
Therefore the shares are;
£280 x 3 / 14 = £60, £280 x 2 / 14 = £40, £280 x 5 / 14 = £100 and £280 x 4 / 14 = £80

If two sets of numbers have the same ratio, then the sum of multiples of each have this same ratio.

Example 15 : 5 is the same ratio as 3 : 1
Therefore the ratio (2 x 15 + 3 x 3) : (2 x 5 + 3 x 1) is the same as 3 : 1
The values inside the brackets are 39 : 13 which is 3 : 1

Squares and Square Roots

The Square of a number is the number times itself. The Square Root of a number is the inverse of this.

Examples
The Square of 12 is $12 \times 12 = 144$. This is written as $12^2 = 144$
The Square Root of 36 is 6. This is written as $\sqrt{36} = 6$

Useful values to remember are;
$\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$ and $\sqrt{10} \approx 3.162$
Also $1/\sqrt{2} = \sqrt{2}/2 \approx 0.707$
The symbol $\approx$ means approximate value.

Cubes and Cube Roots

Similarly, the cube of a number is the number times itself and times itself again
The Cube Root is the inverse and is written $\sqrt[3]{\ }$. Thus $\sqrt[3]{27} = 3$

Indices

The square of 8 is 8×8 and is written 8^2, ie the index is 2.
Alternatively it is said that 8 is raised to the power of 2
Similarly $8 \times 8 \times 8 \times 8 \times 8$ is written 8^5, ie the index is 5, or 8 is raised to the power of 5.

A useful value to remember is $2^{10} = 1024$. This is the number called a kB for computers.

Index value 1

Any number raised to the power of 1 has the value of the number
Example $6^1 = (6) = 6$

Negative Indices

$8^2 \times 8^3 = (8 \times 8) \times (8 \times 8 \times 8) = 8^5$
When two factors are the same number with indices, add the indices

Similarly $8^3 \times (1/8) = (8 \times 8 \times 8) / (8) = 8^2$
Thus $1/8$ behaves as 8^{-1}
Thus a negative index is the same as the reciprocal (1)

Example Evaluate 5^{-3}
$5^{-3} = 1 / 5^3 = 1 / 125 = 0.008$

Zero Index

Any number raised to the power of 0 has the value 1 (2)
Example $4^{-1} \times 4 = 4^0$ but $4^{-1} \times 4 = 1/4 \times 4 = 1$ Thus $4^0 = 1$

Fractional Indices

$5^{1/2} \times 5^{1/2} = 5^1 = 5$ There fore $5^{1/2} = \sqrt{5}$ (3)
Similarly $7^{1/3} = \sqrt[3]{7}$

Exponentials

$100 = 10^2$
$1000 = 10^3$ etc
These are called exponentials of 10

Thus 5.67 E 3 means $5.67 \times 10^3 = 5{,}670$
And 5.67 E −3 means $5.67 \times 10^{-3} = 0.00567$
Numbers in this form are said to be in Scientific Notation

Logarithms (Logs) to base 10

The Logarithm of a number to base 10 is the index of 10 to equal the number

$\text{Log}_{10}\,100 = 2$
since 10 raised to the power of 2 equals 100

Similarly $\text{Log}_{10}\,3.162 \approx 0.5$
Since $10^{0.5} = \sqrt{10} \approx 3.162$
$\text{Log}_{10}\,10 = 1$

$2 \times \text{Log}_{10}\,10 = 2 = \text{Log}_{10}\,100$
Similarly $5 \times \text{Log}_{10}\,10 = \text{Log}_{10}\,10^5 = \text{Log}_{10}\,100{,}000$

$\text{Log}_{10}100 + \text{Log}_{10}1000 = 2 + 3 = 5 = \text{Log}_{10}(100{,}000)$
Therefore $\text{Log}_{10}100 + \text{Log}_{10}1000 = \text{Log}_{10}(100 \times 1000)$

Similarly;
$\text{Log}_{10}4 + \text{Log}_{10}6 = \text{Log}_{10}(4 \times 6) = \text{Log}_{10}24$
$\text{Log}_{10}6 - \text{Log}_{10}4 = \text{Log}_{10}(6 / 4) = \text{Log}_{10}1.5$
$\text{Log}_{10}6 - \text{Log}_{10}6 = \text{Log}_{10}(6 / 6)$ Therefore $\text{Log}_{10}1 = 0$

Decimal and other number systems

Decimal System	0 1 2 3 4 5 6 7 8 9 10 11 12 etc
Binary System	0 1 10 11 100 101 110 111 etc
Octal System	0 1 2 3 4 5 6 7 10 11 etc
Hexadecimal System	0 1 2 3 4 5 6 7 8 9 A B C D E F 10 11 etc

Traditional Arithmetic uses the decimal system. Computers use the binary system (switches are on or off, only two states). Octal and hexadecimal systems are closely related to the binary system.

The decimal value N of an octal number 1234 is;
N = (4) + (3) x 8 + (2) x 8 x 8 + (1) x 8 x 8 x 8 = 692
where the numbers in brackets are the digits of the octal number

To convert a decimal number to octal, divide the number by 8 and the remainder is the last digit. Divide the factor by 8 and the remainder is the next to last digit etc.

Example Decimal 69 = Octal 105 (ie $69 = 1 \times 8^2 + 0 \times 8 + 5$)

2 ALGEBRA

Algebraic Symbols

Letters are used to denote unspecified values. A value can be assigned to the letter later after calculations, or calculations can be used to find out the value of the letter.

Letters A, a, B, b etc are usually used for values which remain constant throughout the calculations and *x, y, z* etc are usually used for values which may change.

Multiplication and Division

a multiplied by b is written as ab
a multiplied by a $= a^2$ (4)

a^m multiplied by $a^n = a^{m+n}$ (5)

$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$ (6)

$(a^m)^n = a^{mn}$ (7)

a divided by b $= a/b = ab^{-1}$ (8)

Example 1 on long division. Divide $(a\,x^2 + b\,x + c)$ by $(x - 1)$

$$
\begin{array}{r|l}
 & \underline{\quad a x + (a + b)} \\
x - 1 & a x^2 + b x \quad + \quad c \\
 & \underline{a x^2 - a x} \\
 & (a + b)\,x \quad + \quad c \\
 & \underline{(a + b)\,x - (a + b)} \\
 & \qquad\qquad a + b + c
\end{array}
$$

Therefore;

$(a\,x^2 + b\,x + c) / (x - 1) \;=\; a\,x + a + b + \;(a + b + c) / (x - 1)$

Check the answer

$$
\begin{aligned}
& a\,x + a + b + \;(a + b + c) / (x - 1) \\
&= ((a\,x + a + b)\,(x - 1) + a + b + c) / (x - 1) \\
&= (a\,x^2 + a\,x + b\,x - a\,x - a - b + a + b + c) / (x - 1) \\
&= (a\,x^2 + b\,x + c) / (x - 1)
\end{aligned}
$$

Example 2 on long division. Divide $(8x^3 - 1)$ by $(2x + 1)$

Insert missing terms with co-efficient shown as zero

$$
\begin{array}{r|l}
 & \underline{\;4x^2 - 2x + 1} \\
2x + 1 & 8x^3 + 0x^2 + 0x - 1 \\
 & \underline{8x^3 + 4x^2} \\
 & \quad - 4x^2 + 0x \\
 & \quad \underline{- 4x^2 - 2x} \\
 & \qquad\quad 2x \; - 1 \\
 & \qquad\quad \underline{2x + 1} \\
 & \qquad\qquad - 2
\end{array}
$$

Ans $(8x^3 - 1) / (2x + 1) = (4x^2 - 2x + 1) - \;2/ (2x + 1)$

Factors

Let $F(x)$ be a function with a factor $(x - a)$

Then $F(x) = (x - a) \times f(x)$ where $f(x)$ is another function of x

This is true for all values of x. Put $x = a$, then $F(a) = 0 \times f(x) = 0$

Conversely, if a value for x can be found that makes $F(x) = 0$,
then $(x - a)$ is a factor of $F(x)$ (9)

Remainder

Let $F(x)$ be any function of x
Let $F(x) = [A(x)]\,(x - a) + R$
where $[A(x)]$ is another function of x
This is true for all values of x Put $x = a$ to get $F(a) = R$
Therefore $(x - a)$ is a factor of $[F(x) - F(a)]$ (10)

Example
Let $F(x) = 7x^3 - 6x^2 + 8x - 9$ and $a = 2$
$F(a) = 7 \times 8 - 6 \times 4 + 8 \times 2 - 9 = 39$
$F(x) - F(a) = 7x^3 - 6x^2 + 8x - 48 = (x - 2)(7x^2 + 8x + 24)$

Factorizing

Many algebraic expressions can be factorized.
Example

$$x^2 + 3ax - 10a^2 = (x - 2a)(x + 5a)$$

To factorize $A x^2 + B x + C = 0$ where A, B and C are numbers
If AC (ie A times C) is -ive, look for factors of AC whose Sum = B
If AC (ie A times C) is +ive, look for factors of AC whose Difference = ±B

Of special interest

$x^2 - a^2 = (x - a)(x + a)$	Put a = 1	$x^2 - 1 = (x - 1)(x + 1)$	(11)
$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$	Put a = 1	$x^3 - 1 = (x - 1)(x^2 + x + 1)$	(12)
$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$	Put a = 1	$x^3 + 1 = (x + 1)(x^2 - x + 1)$	(13)

Note these all comply with (9) above

Fractions

Algebraic expressions may be fractions. For addition or subtraction, change all fractions to a common denominator, the LCM.

Example
$3/(x^2 - 16) + 5/(x + 4) - 3/(x - 4)$
$= [3 + 5(x - 4) - 3(x + 4)]/(x^2 - 16)$
$= (3 + 5x - 20 - 3x - 12)/(x^2 - 16) = (2x - 29)/(x^2 - 16)$

Partial Fractions

The symbol $\equiv$ is used to show that the expressions are equal for all values of x.

Let $\dfrac{a\,x^2 + b\,x + c}{(x+\alpha)\,(x+\beta)\,(x+\gamma)} \equiv \dfrac{A}{(x+\alpha)} + \dfrac{B}{(x+\beta)} + \dfrac{C}{(x+\gamma)}$

$$\frac{a\,x^2 + b\,x + c}{(x+\beta)\,(x+\gamma)} \equiv A + (x+\alpha)\,[\,B/(x+\beta) + C/(x+\gamma)\,]$$

This is true for all values of x. Therefore put $x = -\alpha$

$$A = \frac{a\alpha^{2} - b\,\alpha + c}{\beta - \alpha)\,(\gamma - \alpha)}$$

B and C can be evaluated by the same method

The fractions $\dfrac{A}{(x+\alpha)}$, $\dfrac{B}{(x+\beta)}$ and $\dfrac{C}{(x+\gamma)}$

are called Partial Fractions of the original function $\dfrac{a\,x^2 + b\,x + c}{(x+\alpha)\,(x+\beta)\,(x+\gamma)}$

In general;

$$\frac{F(x)}{(x+\alpha_1)\,(x+\alpha_2)\,(x+\alpha_3)....} \equiv \frac{A_1}{(x+\alpha_1)} + \frac{A_2}{(x+\alpha_2)} + \frac{A_3}{(x+\alpha_3)} + \,........$$

Where $A_1 = \left|\, \dfrac{F(x)}{(x+\alpha_2)\,(x+\alpha_3)\,(x+\alpha_4).........} \,\right|_{(x+\alpha_1)=0}$ (14)

This expression means put $(x + \alpha_1) = 0$, ie $x = -\alpha_1$, in the expression inside the box

A_2 , A_3 etc can be evaluated the same way.

An alternative method for evaluating A_1 , A_2 etc is to multiply the identity by the LCM and equate co-efficients of x, x^2 etc.

If the Numerator contains x to equal or higher power than the Denominator, divide first and split the remainder into Partial Fractions.

Splitting an expression into Partial Fractions may seem a pointless academic exercise, but we find later that many problems can only be solved this way.

Example Split into Partial Fractions $(x^3 + 1) / [(x - 2) (x - 3)]$
The denominator $(x - 2) (x - 3) = x^2 - 5x + 6$
Make x times the denominator the first term in the numerator

$$\frac{x^3 + 1}{(x - 2)(x - 3)} \equiv \frac{x(x^2 - 5x + 6) + 5x^2 - 6x + 1}{x^2 - 5x + 6}$$

$$\equiv x + \frac{5x^2 - 6x + 1}{x^2 - 5x + 6}$$

$$\equiv x + \frac{5(x^2 - 5x + 6) + 25x - 30 - 6x + 1}{x^2 - 5x + 6}$$

$$\equiv x + 5 + \frac{19x - 29}{x^2 - 5x + 6}$$

$$\equiv x + 5 + A/(x - 2) + B/(x - 3)$$

where $A = [(19x - 29) / (x - 3)]_{x - 2 = 0} = (38 - 29) / (2 - 3) = -9$
and $B = [(19x - 29) / (x - 2)]_{x - 3 = 0} = (57 - 29) / (3 - 2) = 28$

Thus $(x^3 + 1)/(x - 2)/(x - 3) \equiv x + 5 - 9 / (x - 2) + 28 / (x - 3)$

If the Denominator does not factorize completely, the Partial Fraction with denominator containing x^n must have a numerator containing x^{n-1} , x^{n-2} ... etc

Example Split into Partial Fractions $1/(x^3 - 1)$

$$1/(x^3 - 1) \equiv 1/[(x - 1) (x^2 + x + 1)]$$
$$\equiv A/(x - 1) + (Bx + C)/(x^2 + x + 1)$$

$$A = [1/(x^2 + x + 1)]_{x - 1 = 0} = 1/(1 + 1 + 1) = 1/3$$

Multiply the expressions by $(x^3 - 1)$
$$1 \equiv A(x^2 + x + 1) + (Bx + C)(x - 1)$$
$$\equiv Ax^2 + Ax + A + Bx^2 + Cx - Bx - C$$

This is true for all values of x. Therefore the coefficients of x can be equated.
Coefficient of x^2 $\quad 0 = A + B$
Coefficient of x $\quad 0 = A + C - B$
Constant term $1 = A - C$

Hence $A = 1/3 \qquad B = -1/3 \qquad$ and $C = -2/3$

If the Denominator contains two equal factors, the method fails unless the two factors are treated as one factor.

$$\frac{F(x)}{(x-\alpha)^2(x-\beta)} \equiv \frac{Ax+B}{(x-\alpha)^2} + \frac{C}{(x-\beta)}$$

But $\dfrac{Ax+B}{(x-\alpha)^2} = \dfrac{Ax - A\alpha + A\alpha + B}{(x-\alpha)^2} = \dfrac{A}{(x-\alpha)} + \dfrac{A\alpha + B}{(x-\alpha)^2}$

Thus;

$$\frac{F(x)}{(x-\alpha)^2(x-\beta)} \equiv \frac{A}{(x-\alpha)} + \frac{B'}{(x-\alpha)^2} + \frac{C}{(x-\beta)} \qquad (15)$$

where $B' = [\,F(x)/(x-\beta)\,]_{x-\alpha=0}$ and $C = [\,F(x)/(x-\alpha)^2]_{x-\beta=0}$
and A is found by equating coefficients

Ratios

Let $a/b = c/d \qquad$ Put $c = \alpha\, a \qquad$ therefore $d = c\,b/a = \alpha\, b$
Then $(ma + nc)/(mb + nd) = (m + n\alpha)/(m + n\alpha)\; a/b = a/b$
ie if $a/b = c/d$ then $(ma + nc)/(mb + nd) = a/b$ (16)

Conversely
if $a/b = (ma + nc)/(mb + nd)$ then $c/d = a/b$ (17)

Irrational Functions

Irrational Functions are functions containing square root, cube root etc.
To move a square root from the denominator to the numerator of a fraction.
Use Eqtn (11) $\qquad$ ie $(x - a)(x + a) = (x^2 - a^2)$

Example Put the irrational term of $1/(a + \sqrt{b})$ in the numerator

Multiply top and bottom by $(a - \sqrt{b})$
Thus

$$\frac{1}{a+\sqrt{b}} = \frac{(a-\sqrt{b})}{(a+\sqrt{b})\,.\,(a-\sqrt{b})} = \frac{a-\sqrt{b}}{a^2-b} \qquad (18)$$

The irrational term has been moved to the numerator

Equations

An equation is a statement that two expressions are equal.
The expressions contain unknowns (eg x, y etc)

$x - 6 = 0$ is an equation. The equation is satisfied if and only if $x = 6$.

$x^2 + x - 12 = 0$ is also an equation. The equation is satisfied if $x = 3$ or $x = -4$.
In this equation, $x^2 + x - 12$ can be factorised to $(x + 4)\,(x - 3)$
Thus the equation can be written $(x + 4)\,(x - 3) = 0$

This equation is satisfied if either factor equals zero.
There are two factors so there are two solutions.

In general, if there are n factors containing x, there are n solutions.
But with n factors containing x, then the highest power of x is x^n.
Conversely, if x^n is the highest power of x (after rationalising), then there are n solutions (19)

If n + 1 solutions can be found for an equation with n the highest power of x,
then the equation is an identity, ie is satisfied for all values of x.

Consider the quadratic equation;

$$A x^2 + B x + C = 0 \text{ with solutions } x = \alpha_1 \text{ and } x = \alpha_2$$

The equation can be written $A(x - \alpha_1)(x - \alpha_2) = 0$

Hence $A[x^2 - (\alpha_1 + \alpha_2) x + \alpha_1\alpha_2] = 0$

The equations are identical, so the coefficients are the same

Coefficient of x $\quad (\alpha_1 + \alpha_2) = -B/A$

Constant term $\quad \alpha_1\alpha_2 = C/A$

Consider now the cubic equation;

$A x^3 + B x^2 + C x + D = 0$ with solutions $x = \alpha_1$, $x = \alpha_2$ and $x = \alpha_3$

The equation can be written $A(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = 0$

Therefore $A[x^2 - (\alpha_1 + \alpha_2) x + \alpha_1\alpha_2](x - \alpha_3) = 0$

$$A[x^3 - (\alpha_1 + \alpha_2) x^2 + \alpha_1\alpha_2 x - \alpha_3 x^2 + (\alpha_1\alpha_3 + \alpha_2\alpha_3)x - \alpha_1\alpha_2\alpha_3] = 0$$

$$A[x^3 - (\alpha_1 + \alpha_2 + \alpha_3) x^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) x - \alpha_1\alpha_2\alpha_3] = 0$$

The Greek letter sigma $\sum$ is used to mean "The Sum of terms like"

$\sum(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3$

$\sum(\alpha_1\alpha_2) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$

Thus the equation is $A[x^3 - \sum(\alpha_1) x^2 + \sum(\alpha_1\alpha_2) x - \alpha_1\alpha_2\alpha_3] = 0$

Equating coefficients;

$$\sum(\alpha_1) = -B/A$$
$$\sum(\alpha_1\alpha_2) = C/A$$
$$\alpha_1\alpha_2\alpha_3 = -D/A$$

The process can be repeated for higher powers of x and the same pattern will be found.
For a quartic, there will be an additional coefficient $\sum(\alpha_1\alpha_2\alpha_3)$
and the constant term will be $\alpha_1\alpha_2\alpha_3\alpha_4$

In general for an equation;

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + A_{n-3} x^{n-3} \ldots + A_1 = 0$$

The equation can be written;

$$A_n (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots\ldots\ldots\ldots (x - \alpha_n) = 0$$

Equating coefficients;

$$\sum(\alpha_1) = -A_{n-1}/A_n \qquad (20)$$

$$\sum (\alpha_1 \alpha_2) = A_{n-2} / A_n$$
$$\sum (\alpha_1 \alpha_2 \alpha_3) = -A_{n-3} / A_n$$
$$\sum (\alpha_1 \alpha_2 \alpha_3 \ldots\ldots \alpha_r) = (-1)^r A_{n-r} / A_n \tag{21}$$
$$\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n A_1 / A_n \tag{22}$$

$\alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_n$ are called the roots of the equation $F(x) = 0$
and $x = \alpha_1$, $x = \alpha_2$, $x = \alpha_3$ etc are the solutions

Example
If α, β and γ are the roots of $x^3 + a x^2 + b x + c = 0$, find the value of $\alpha^3 + \beta^3 + \gamma^3$

$(\alpha + \beta + \gamma)^3 = (\alpha + \beta + \gamma)(\alpha + \beta + \gamma)(\alpha + \beta + \gamma)$
Multiplying out gives
ie $[\sum(\alpha)]^3 = \sum(\alpha^3) + 3\sum(\alpha^2 \beta) + 6\alpha\beta\gamma$
But
$[\sum(\alpha)][\sum(\alpha\beta)] = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \alpha\gamma) = \sum(\alpha^2\beta) + 3\alpha\beta\gamma$
Therefore $\sum(\alpha^3) = [\sum(\alpha)]^3 - 3\sum(\alpha^2\beta) - 6\alpha\beta\gamma$
$= [\sum(\alpha)]^3 - 3[\sum(\alpha)][\sum(\alpha\beta)] + 9\alpha\beta\gamma - 6\alpha\beta\gamma$
$= [\sum(\alpha)]^3 - 3[\sum(\alpha)][\sum(\alpha\beta)] + 3\alpha\beta\gamma$
But $\sum(\alpha) = -a$, $\sum(\alpha\beta) = b$ and $\alpha\beta\gamma = -c$
Therefore $\sum(\alpha^{\square}) = -a^3 + 3ab - 3c$

Graphical Solution

To solve the equation $f(x) = 0$
If possible, factorize $f(x)$ to the form;
$f(x) = a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots (x - \alpha_n)$
The solutions are then $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$

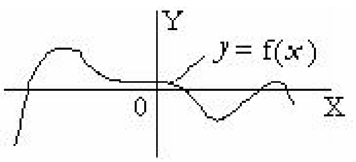


Figure 12: Graphical Solution

However it is often not possible to factorize $f(x)$.
Solutions can be obtained by plotting the curve $y = f(x)$

Choose a value for x.
Substitute this value for x in $f(x)$ to obtain a value for y.
Mark a point on graph paper at the values of x and y.
Repeat again and again choosing new values for x. Join up the points to obtain a curve.
The solutions are the values of x where the curve crosses the OX axis, ie where $y = 0$.
For a more accurate solution, repeat at a larger scale over a small range by each solution.

Solution by Computer

It is easy to write a program that automatically chooses a value for x, calculates the value for y and compares it with the previous value. If the sign is different, then there is a solution between the two values. The step between the x values can be reduced and the process repeated between the two previous values. The process can be repeated until the required accuracy is obtained for this solution. The program should then look for another solution etc.

Real and Complex Solutions

Solution by the Graphical method or Computer finds all points where the curve crosses the OX axis. This is all that is needed for practical problems. However there may be other "hidden" solutions.

Consider the equation $x^2 + 1 = 0$
Plot $y = x^2 + 1$ and the curve never crosses the OX axis.
The highest power of x is 2, so there should be 2 solutions.

The answer lies in an imaginary value, the square root of (-1). It does not exist in the real world.
This imaginary value is called "eye" and denoted by the lower case letter i.
Thus $i^2 = -1$

The equation $x^2 + 1 = 0$ can be written $(x + i)(x - i) = 0$
The two solutions are $x = -i$ and $x = i$

Solutions that contain i are called Complex Solutions.
Typically they are in the form $(a + i\,b)$ and $(a - i\,b)$
Complex solutions to a Real equation always appear in pairs in this form, called a Conjugate Pair.
The Product of a Conjugate Pair gives a Real result, since;

$$(a + i\,b)(a - i\,b) = a^2 + b^2 \qquad (23)$$

Quadratic Equations

A quadratic equation has x to the power of 2 and therefore two solutions.

The general form of a quadratic is;

$$A x^2 + B x + C = 0$$

Divide by A and rearrange $\quad x^2 + (B/A)\,x = -C/A$
Add $B^2/(4A^2)$ to both sides

$$x^2 + (B/A)\,x + B^2/(4A^2) = B^2/(4A^2) - C/A$$

$$[x + B/(2A)]^2 = [1/(4A^2)]\,(B^2 - 4AC)$$

Take square roots $\quad x + B/(2A) = \pm 1/(2A)\,\sqrt{(B^2 - 4AC)}$
Thus the solution to the quadratic is

$$x = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A} \qquad (24)$$

If	$B^2 > 4AC$	then there are two Real Solutions	(25)
If	$B^2 = 4AC$	then there are two equal solutions	(26)
If	$B^2 < 4AC$	then there are two Complex Solutions	(27)

If $y = Ax^2 + Bx + C$ is plotted, the solutions to the equation $Ax^2 + Bx + C = 0$ are at the intersection of the curve with the line $y = 0$, the line OX in the diagrams.

The diagrams show the three different cases.

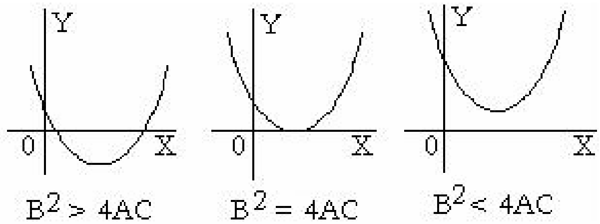


Figure 13: Quadratic equation

If the quadratic is rearranged to the form $x^2 + 2Bx + C = 0$,
then the solution is $x = -B \pm \sqrt{(B^2 - C)}$ (28)

Inverse Functions

Let $f(x, y) = 0$ Solve for x in terms of y
$x = g(y)$ where $g(y)$ means a function of y
Similarly $y = h(x)$ where $h(x)$ means a function of x
$g(y)$ and $h(x)$ are related by the original equation
$g(x)$ and $h(x)$ are said to be <u>inverse functions</u> of each other
$y = g(x)$ is the original equation with x and y interchanged
Plot $g(x)$ against x, and $h(x)$ against x
The curves are mirror images about the line $y = x$

Simultaneous Equations

If there are two equations and two variables, they are called simultaneous equations.

$F_1(x,y) = 0$ and $F_2(x,y) = 0$ (29)

1) Graphical Solution
   Plot the two curves and the intersections give the solutions.

2) Theoretical Solution
   Eliminate one variable and solve for the other.
   Substitute the second variable in either equation to evaluate the first variable

3) Numerical Solution, two variables (for solution by computer)
   Choose a value for one variable by a FOR...TO loop
   Chose a value for the second variable by a FOR...TO loop nested in the first loop
   Evaluate both functions and if the result is a better match than any previous result, record the value of both variables.

Example on Simultaneous Equations

$$2x + 3y = 7$$

$$5x + 2y = 9$$

(1) Graphical Solution

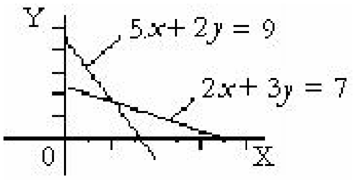


Figure 14: Simultaneous Equations

Plot $y = (7 - 2x) / 3$

And $y = (9 - 5x) / 2$

The solution is the values of x and y where the lines (or curves) cross

(2) Theoretical Solution

Eliminate x
5 x eqtn(1) – 2 x eqtn(2) gives;
$10x + 15y - (10x + 4y) = 35 - 18$
Thus $11y = 17$
Thus $y = 17 / 11 = 1.545$
Therefore $x = (7 - 3 \times 1.545) / 2 = 1.183$

Simultaneous Equations, n variables

If there are n variables and n different simultaneous equations connecting them, then there are solutions for all the variables (although the solutions may be Complex Solutions).

If there are only n – 1 equations, then the relationship between any two variables can be plotted as a curve.

Example on simultaneous quadratic equations (two variables)

$$E1 = 5x^2 + 3x - 4y^2 + 4y - 1.544 = 0$$
$$E2 = 2x^2 - 8x - y^2 + 4y - 4.744 = 0$$

Find solutions between $x = -100$ and $x = 100$

Eliminate y^2, subtract 4 times E2 from E1

$$5x^2 - 8x^2 + 3x + 32x - 4y^2 + 4y^2 + 4y - 16y - 1.544 + 18.976 = 0$$
$$-3x^2 + 35x - 12y + 17.434 = 0$$
$$y = (-3x^2 + 35x + 17.434)/12$$

Choose values for x and evaluate E1 and E2
This is best done by computer, (eg by the QBasic program below)

```
CLS: FOR X = -100 TO 100 STEP 10
Y = (-3*X*X + 35*X + 17.434)/12
E1 = 5*X*X + 3*X - 4*Y*Y + 4*Y - 1.544
E2 = 2*X*X - 8*X - Y*Y + 4*Y - 4.744
PRINT "X = "; X; " Y = "; Y; TAB(40); "E1 = ";E1; TAB(60); "E2 = "; E2
NEXT X: END
```

This shows that E1 and E2 change sign between $x = 0$ and $x = 10$ and again between $x = 10$ and $x = 20$.
There are solutions where E1 and E2 change sign

Change the first line of the program to; CLS: FOR X = -10 TO 10
Run the program again and there is a low point between –1 and 0

and there is a solution between $x = 7$ and $x = 8$

Similarly change the first line to; CLS: FOR X = 10 TO 20
and find there is a solution between $x = 16$ and $x = 17$

Repeat with STEP of 0.1 and again with STEP 0.01 between ever narrower limits for x to find the real solutions;

$x = 7.53$ $y = 9.24$
$x = 16.45$ $y = 18.22$

In addition, there is a low point at $x = -0.32$ and $y = 0.49$ indicating complex solutions near here

This method can be used to solve simultaneous equations with two variables provided that at least one of the variables is not higher power than 3. (Eliminate the cubic term and solve the quadratic)

3 GEOMETRY

Angles, Degrees

Angles can be measured in Degrees

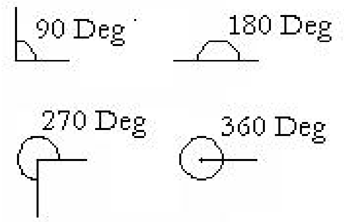


Figure 15; Angles in degrees

A Right Angle is 90 Degrees
Degree is divided into minutes and seconds.
60 minutes = 1 degree
60 seconds = 1 minute

Angles, Radians

Another measurement of angle is the Radian.

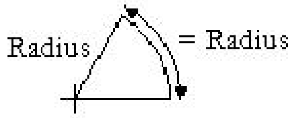


Figure 16: One Radian

One Radian is the angle where the length of the arc is equal to the radius.

$2 \times \pi$ Radians = 360 Degrees (30)
One Radian = approx 57 Degrees

Circles

The circumference of a circle is proportional to the diameter. The ratio is called Pi and is denoted by the Greek letter π

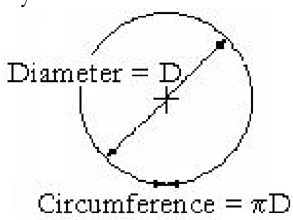


Figure 17: Circumference = πD

The Circumference = π x Diameter = 2 x π x Radius (31)
The value of π is approximately 3.14159. It has been measured to hundreds of digits long, but even then is not exact.

Polygons

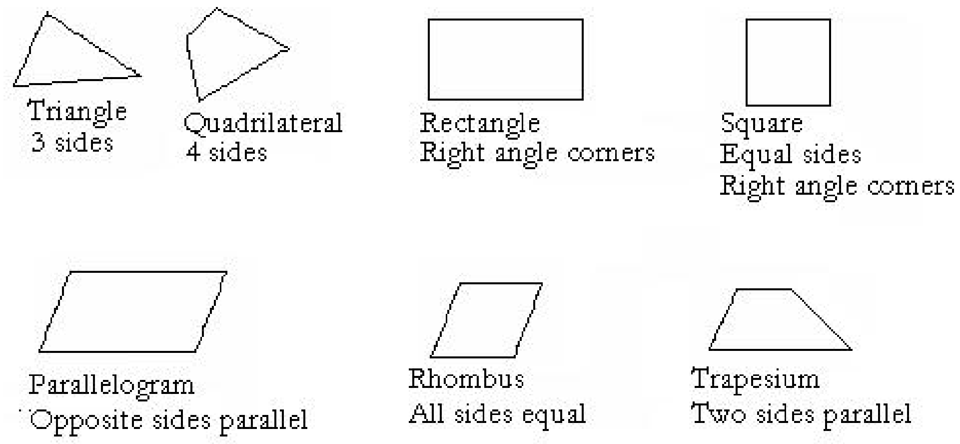


Figure 18: Polygons

Polygon is any enclosed figure with straight sides

Triangle is a three sided polygon
Quadrilateral is a four sided polygon
Rectangle is a quadrilateral with all corners right angles
Square is a rectangle with all sides equal
Parallelogram is a quadrilateral with opposite sides parallel
Rhombus is a parallelogram with all sides equal
Trapezium is a quadrilateral with two parallel sides
Pentagon is a five sided polygon
Regular Pentagon is a pentagon with all sides equal
Hexagon is a six sided polygon

Properties of Angles

Consider a line crossing two parallel lines

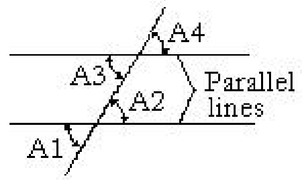


Figure 19: Angles

A1 = A2 (Vertically. Opposite angles)
A2 = A3 (Alternate angles)
A2 = A4 (Corresponding angles) (32)

Angles of a Triangle

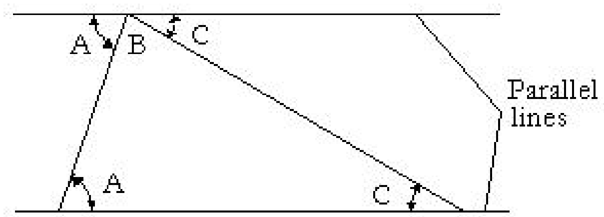


Figure 20: Angles of a triangle

Angles A and A are alternate angles. Angles C and C are alternate angles,
It can be seen that A + B + C = 180 degrees
The Sum of the angles of any triangle equal 180 degrees (33)

Equilateral Triangle

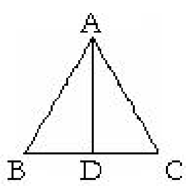


Figure 21: Equilateral triangle
If all three sides of a triangle are equal in length, the triangle is said to be equilateral.

By symmetry, all the angles are equal. But the angles add up to 180 degrees.
Therefore each angle of an equilateral triangle is 60 degrees. (34)

If line AD is drawn perpendicular to BC then angles BAD and CAD are each 30 degrees.

Isosceles Triangles

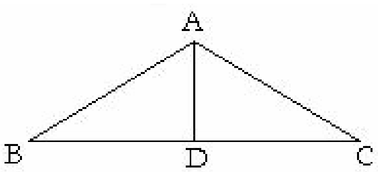


Figure 22: Isoceles triangle

If two of the sides of a triangle are equal, the triangle is said to be an isosceles triangle.
By symmetry, two of the angles are also equal.

The perpendicular AD bisects the angle BAC.

Congruent Triangles

If two triangles are identical, then they are said to be Congruent.
Triangles are congruent if any of the following conditions are met;

Either 1) All three sides are the same on both triangles
Or 2) Two of the sides and the included angle (ie the angle between these sides) are the same
Or 3) Two of the angles and a corresponding side are the same
Or 4) The triangles have a right angle, the side opposite the right angle is equal and one other side . (35)

Note that if two sides and a non included angle are equal, the triangles are not necessarily congruent.

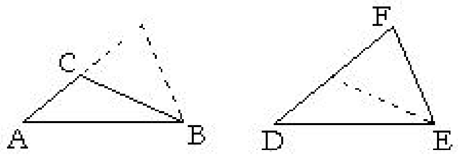


Figure 23: Two sides and non included angle

AB = DE
BC = EF
Angle BAC = Angle EDF

Triangles ABC and DEF are not congruent

Similar Triangles

If two triangles have the same angles but the sides are different, they are said to be similar. The sides are in the same proportion on both triangles. (36)

Pythagoras's Theorem

If the triangle has a right angle, then the side opposite the right angle is called the hypotenuse.

Pythagoras's theorem states;
The square of the hypotenuse equals the sum of the squares of the other two sides.

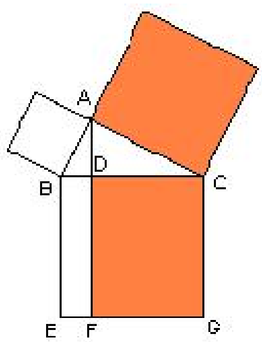


Figure 24: Pythagoras

It can be proved by Euclidian Geometry that the coloured areas of Figure 24 are equal, but the following proof is simpler.

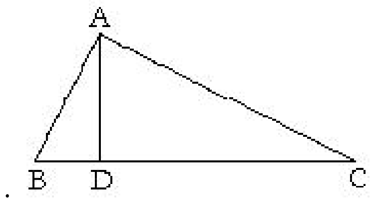


Figure 25: Pythagoras

Triangle ABC has a right angle at BAC
Draw AD perpendicular to BC

Triangles BCA and ACD are similar
since each contains angle ACB and a right angle.

Therefore $AC / BC = DC / AC$
Hence $AC^2 = BC \times DC$

Similarly $AB^2 = BC \times BD$
Add to obtain

$$AB^2 + AC^2 = BC \times DC + BC \times BD$$
$$= BC \times (DC + BD) = BC^2$$

$$AB^2 + AC^2 = BC^2 \qquad (37)$$

Examples of Pythagoras

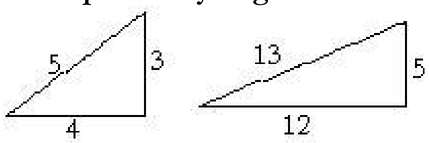


Figure 26: Pythagoras examples

$3^2 + 4^2 = 5^2$
$12^2 + 5^2 = 13^2$

and multiples of these
eg $6^2 + 8^2 = 10^2$

Areas

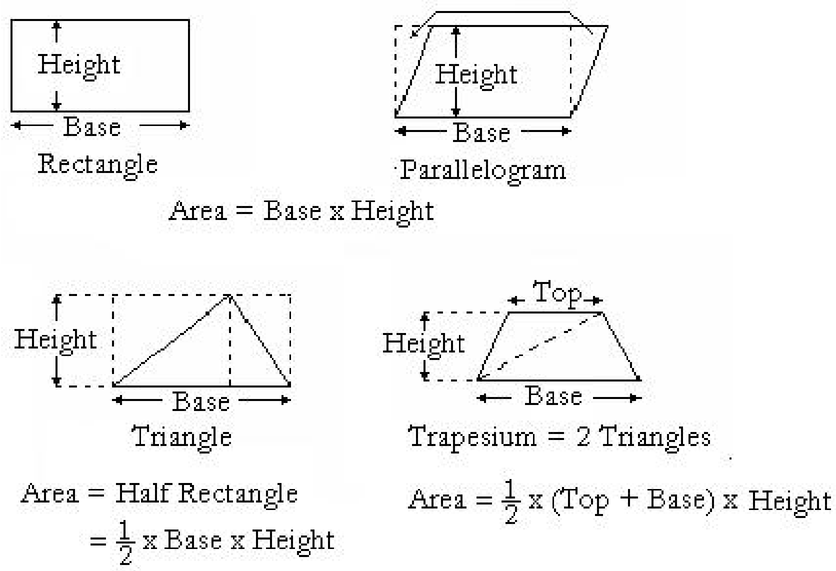


Figure 27: Areas of polygons

Rectangle and Parallelogram	Area = Base x Height	(38)
Triangle	Area = 1/2 x Base x Height	(39)
Trapezium	Area = 1/2 x (Top + Base) x Height	(40)

Area of a Circle

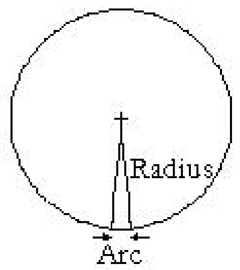


Figure 28: Area of a circle

A Circle can be considered as lots of tiny triangles
Area of each triangle = 1/2 x Arc x Radius

Area of the whole Circle

= 1/2 x Radius x (Sum of arcs)
= 1/2 x Radius x 2 x Pi x Radius
= Pi x $(\text{Radius})^2$

$$\text{Area of a Circle} = \pi R^2 \tag{41}$$

Centre of Area or Centroid

If a geometric shape is cut out of heavy card of uniform thickness and density, it will balance on a point called the Centre of Area or the Centroid.

By Symmetry;
The Centroid of a circle is the centre of a circle.
The Centroid of a rectangle or parallelogram is where the two diagonals cross.

In the Triangle ABC,
AD bisects BC
BE bisects AC
These lines are called Medians

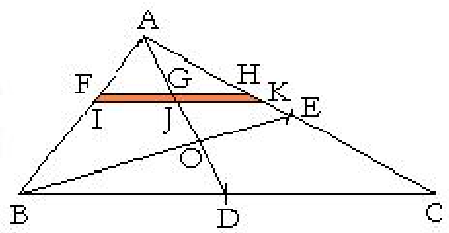


Figure 29: Medians of a Triangle

Triangle AGH is similar to triangle ADC
Therefore GH/AG = DC/AD

Triangle AFG is similar to triangle ABD
Therefore AG/FG = AD/BD

Therefore
GH/AG x AG/FG = DC/AD x AD/BD

GH/FG = DC/BD
But BD = DC, therefore GH = FG
Therefore the thin strip FHKI would balance on a knife edge on GJ
But the whole triangle is made up of strips that balance on the line AD
Therefore the triangle balances on a knife edge on the line AD

Similarly the triangle balances on a knife edge on the line BE
Thus the triangle would balance on a point at O where the two lines cross

All three medians pass through the Centroid which is a unique point. Therefore all three medians meet at this point.

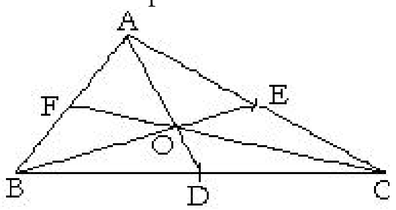


Figure 30: Medians meet at a point

Medians are;
AD, BE and CF
where BD = DC and AE = EC and AF = FB

The Medians (the lines from an apex bisecting the opposite side) all meet at a point

Lines perpendicular to the side and through the opposite apex meet at a point

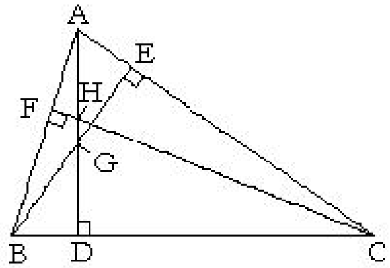


Figure 31: Perpendiculars

In triangle EBC, Angle EBC = 90^0 − angle ACB
In triangle ADC, Angle DAC = 90^0 − angle ACB
Therefore triangles BDG and ADC both contain the same angle and a right angle.

Therefore the triangles are similar.
GD/ BD = DC/ AD
Multiply both sides by BD/ DC
GD/ DC = BD/ AD

Triangle CDH is similar to triangle AFH since they both contain a right angle and the vertically opposite angles CHD and AHF.
But triangle AFH is similar to ADB since they both contain a right angle and the common angle BAD.

Therefore triangles CDH and ADB are similar
BD/ AD = HD/ DC

But GD/DC = BD/AD
Therefore GD/ DC = HD/ DC

Multiply both sides by DC
GD = HD, ie G and H are the same point
This point, where the perpendiculars meet, is called the Orthocentre

Lines bisecting the angles of a triangle all meet at a point

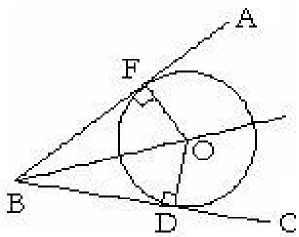


Figure 32: Line bisecting an angle

The circle, centre O, just touches lines BA and BC
OF = OD (radius of the circle)

BF and BD are tangents,
Therefore Angles BFO and BDO are right angles

Triangle BOF is congruent to triangle BOD (hypotenuse and one other side)
Angle ABO = Angle CBO
ie BO bisects Angle ABC

Similarly AO bisects Angle BAC
and CO bisects Angle ACB

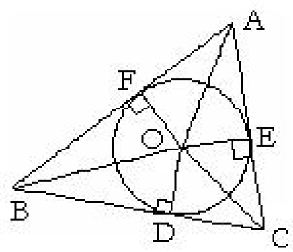


Figure 33: Inscribed circle
Thus the lines bisecting the angles of a triangle all meet at O, the centre of the inscribed circle

Perpendiculars from the mid point of each side all meet at a point

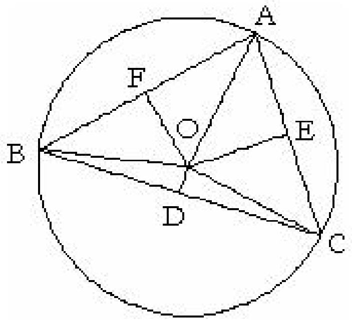


Figure 34: Circumcircle

O is the centre of the circumcircle that passes through the Apexes

OD, OE and OF are perpendicular to sides BC, AC and AB

Triangle BOF is congruent to Triangle AOF (OF is common and hypotenuse AO = BO)

Therefore BF = AF
ie OF is perpendicular to AB and passes through its mid point

Similarly OD is perpendicular to BC and passes through its mid point
Similarly OE is perpendicular to AC and passes through its mid point

Tangent to a circle

The line AB is a tangent to the circle it just touches the circle at T

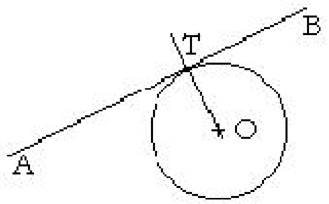


Figure 35: Tangent **to a ircle**

The line is parallel to the circumference where it touches. But the circumference is perpendicular to the radius, so the tangent is perpendicular to the radius OT

Circle touching two lines

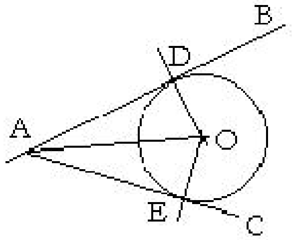


Figure 36: Circle touching two lines

The circle, centre O, touches two lines AB and AC.
The line AO bisects angle BAC since
triangle AOD is congruent to triangle AOE,
(hypotenuse and one other side)

A circle of any radius can touch the two lines

Circle touching three lines

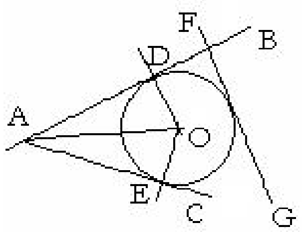


Figure 37: Circle touching three lines.

The circle touches lines AB, AC and FG

There is one only circle that can do this

Circle through two or three points

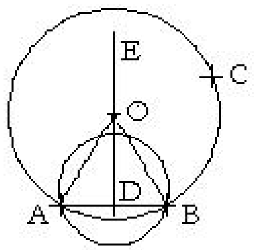


Figure 38: Circle through two points

Figure 38 shows two circles through A and B.
A circle of any diameter greater than the length AB can pass through points A and B.

Sphere touching three or four points

A sphere of any radius above a minimum can touch three points. Imagine a tennis ball or a football balanced on three points.

Only one of the spheres can touch a fourth point that is not in the same plane.

If the four points are in the same plane, the radius of the sphere that touches them all would be infinite.

Sphere touching three or four planes

A sphere of any radius can touch two planes ABCD and ABEF.
Its centre lies on the plane ABGH that bisects ABCD and ABEF.

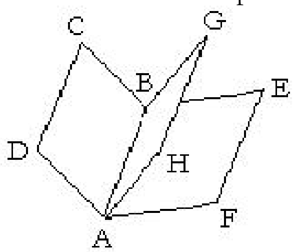


Figure 39: Sphere touchin two planes

If the sphere touches a third plane, its centre lies on the line where the two bisecting planes cross.

One and only one of the spheres will touch a fourth plane.

Nesting Circles

The internal angles of an equilateral triangle are all equal. They add to 180 degrees, and therefore are each 60 degrees. Thus six triangles will exactly fit in a circle that has a radius equal to the side of the triangles.

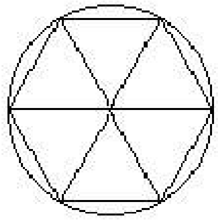


Figure 40: Six equilateral triangles in a circle.

Thus six circles with half the radius and centres at the apexes will exactly fit round a similar circle at the centre.

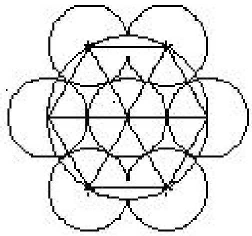


Figure 41: Six circles round a central circle.

Thus seven cores of a multicore cable nest with the outer cores on a pitch circle radius 2R where R is the radius of each core. Hence multicore cables are often seven core

A further twelve cores can be added on a pitch circle radius 4R giving a total of nineteen cores, another popular arrangement.

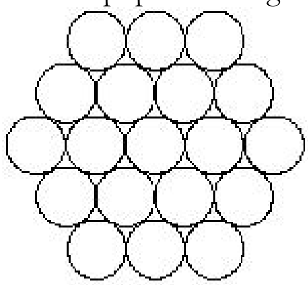


Figure 42: Nineteen cores

4 TRIGONOMETRY

Definitions

Triangle ABC has a right angle at C and sides with lengths a, b and c

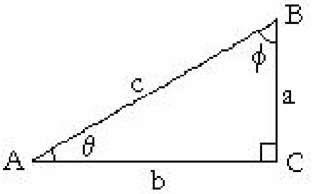


Figure 43: Trigonometrical functions

Trigonometrical Functions are defined as;
Sine θ = a / c
Cosine θ = b / c
Tangent θ = a / b
Cosecant θ = c / a
Secant θ = c / b
Cotanangent θ = b / a

These are usually shortened to
Sin θ = a/c   Cos θ = b/c   Tan θ = a/b
Cosec θ = c/a   Sec θ = c/b   Cot θ = b/a   (42)

The Inverse of Sin θ is called Arc Sin θ or $\text{Sin}^{-1}\,\theta$

In the diagram;
Arc Sin (a/c) = θ Arc Cos (b/c) = θ and Arc Tan (a/b) = θ.   (43)
Alternatively, Sin^{-1} (a/c) = θ, Cos^{-1} (b/c) = θ and Tan^{-1} (a/b) = θ
Write $\text{Sin}^2\,\theta$ to mean $(\text{Sin}\,\theta)^2$ or (Sin θ) x (Sin θ),
but to avoid confusion, do not use $\text{Sin}^{-1}\,\theta$ to mean 1 / Sin θ

Formulae connecting Trig Functions

(Sin θ)/(Cos θ) = [a/c] / [b/c] = a/b = Tan θ (44)
Similarly
Cosec θ = 1 / Sin θ (45)
Sec θ = 1 / Cos θ (46)
Cot θ = 1 / Tan θ (47)

$\text{Sin}^2\,\theta + \text{Cos}^2\,\theta = a^2 / c^2 + b^2 / c^2 = (a^2 + b^2) / c^2 = c^2 / c^2 = 1$

Thus $\text{Sin}^2\,\theta + \text{Cos}^2\,\theta = 1$ (48)

Divide by Cos^2 $\text{Tan}^2\,\theta + 1 = \text{Sec}^2\,\theta$ (49)

Divide by $\text{Sin}^2\,\theta$ $1 + \text{Cot}^2\,\theta = \text{Cosec}^2\,\theta$ (50)

The angles of a triangle add to 180^0 Therefore $\phi = (90^0 - \theta)$

$\text{Sin}\,(90^0 - \theta) = b / c = \text{Cos}\,\theta$ (51)

$\text{Cos}\,(90^0 - \theta) = \text{Sin}\,\theta$ (52)

$\text{Tan}\,(90^0 - \theta) = \text{Cot}\,\theta$ (53)

$\text{Cot}\,(90^0 - \theta) = \text{Tan}\,\theta$ (54)

Particular values of Trig Functions

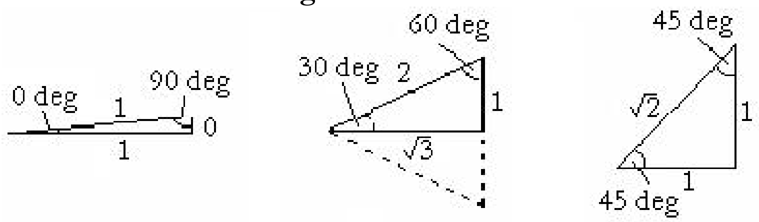


Figure 44: Particular values

Table 2: Particular values

Angle	0 Deg	30 Deg	45 Deg	60 Deg	90 Deg
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
Cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
Tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	Infinity

(55)

Small angles

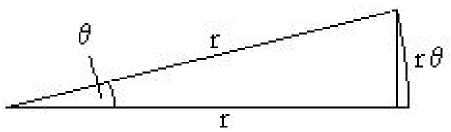


Figure 45: Small angles

If θ is small and in radians

$\text{Sin}\,\theta \approx r\,\theta / r \approx \theta$ (56)

$\text{Tan}\,\theta \approx r\,\theta / r \approx \theta$ (57)

From (48), $\quad \mathrm{Cos}\,\theta = \sqrt{(1 - \mathrm{Sin}^2\,\theta)} \approx \sqrt{(1 - \theta^2)}$

But $\quad (1 - \tfrac{1}{2}\,\theta^2)^2 = 1 - \tfrac{1}{2}\,\theta^2 - \tfrac{1}{2}\,\theta^2 + \tfrac{1}{4}\,\theta^4 \approx 1 - \theta^2$ since θ is small

Therefore $\quad \mathrm{Cos}\,\theta \approx (1 - \tfrac{1}{2}\,\theta^2)$ (58)

Angles over 90 degrees

Angle between 90 and 180 degrees

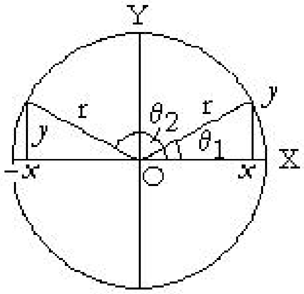


Figure 46: Angle between 90 and 180 degrees

$\mathrm{Sin}\,\theta_2 = y/\mathrm{r} = \mathrm{Sin}\,\theta_1$

$\mathrm{Cos}\,\theta_2 = -x/\mathrm{r} = -\mathrm{Cos}\,\theta_1$

$\mathrm{Tan}\,\theta_2 = y/(-x) = -\mathrm{Tan}\,\theta_1$

Angle between 180 and 270 degrees

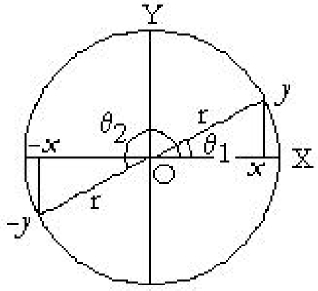


Figure 47: Angle between 180 and 270 degrees
Sin $\theta_2 = -y/\mathrm{r} = -$ Sin θ_1
Cos $\theta_2 = -x/\mathrm{r} = -$ Cos θ_1
Tan $\theta_2 = -y/(-x) =$ Tan θ_1

Angle between 270 and 360 degrees

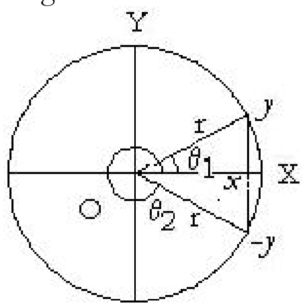


Figure 48: Angle between 270 and 360 degrees
Sin $\theta_2 = -y/\mathrm{r} = -$ Sin θ_1
Cos $\theta_2 = x/\mathrm{r} =$ Cos θ_1
Tan $\theta_2 = -y/x = -$ Tan θ_1

Negative Angles

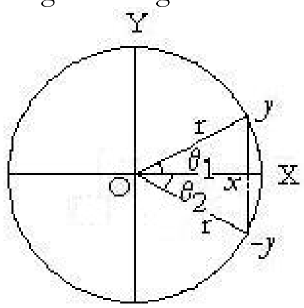


Figure 49: Negative angles
Sin $(-\theta_2) = -y/\mathrm{r} = -$ Sin θ_1
Cos $(-\theta_2) = x/\mathrm{r} =$ Cos θ_1
Tan $(-\theta_2) = -y/x = -$ Tan θ_1
These are still true for angles larger than 90 degrees

Angles in radians plus or minus $2n\pi$ where n is an integer
$\text{Sin}\ (\pm\ 2n\pi + \theta_1) = \text{Sin}\ \theta_1$
$\text{Cos}\ (\pm\ 2n\pi + \theta_1) = \text{Cos}\ \theta_1$
$\text{Tan}\ (\pm\ 2n\pi + \theta_1) = \text{Tan}\ \theta_1$

The word CAST is a memory aid.

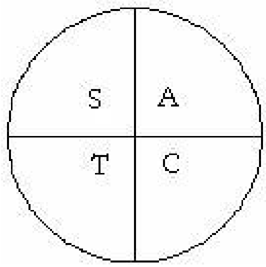


Figure 50: CAST
C = Cos positive
A = All positive
S = Sin positive
T = Tan positive

Sin (A + B) etc

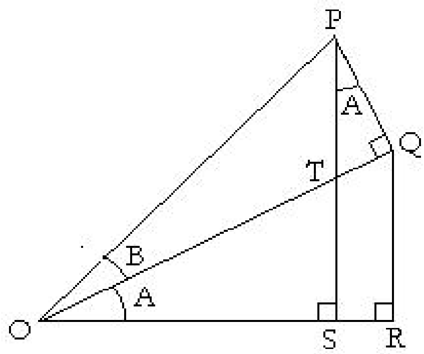


Figure 51: Sin (A + B)
Sin (A + B) = PS / OP
PS = PT + TS
= PQ x Sec A + (OQ – TQ) x Sin A
= PQ/Cos A + OQ x Sin – TQ x Sin A
= PQ/Cos A + PQ x Sin A /Tan B – PQ x Sin A x Tan A
= PQ x [(1 – Sin^2 A) / Cos A + Sin A x Cos B / Sin B
= PQ x (Cos A x Sin B + Sin A x Cos B) / Sin B

But Sin B = PQ / OP
PS = OP x (Cos A x Sin B + Sin A x Cos B)

Therefore Sin (A + B) = Sin A x Cos B + Cos A x Sin B
Writing Sin A Cos B to mean (Sin A) x (Cos B) etc
Sin (A + B) = Sin A Cos B + CosA SinB

Put B = – C
Sin (A – C) = Sin A Cos C – Cos A Sin C

Also OS = (OQ – QT) Cos A
= (PQ / Tan B – PQ x Tan A) x Cos A
= OP x Sin B (Cos A Cos B / Sin B – Sin A)
But OS/OP = Cos (A + B)
Therefore Cos (A + B) = Cos A Cos B – Sin A Sin B

Put B = – C
Cos (A – C) = Cos A Cos C + Sin A Sin C
Tan (A + B) = Sin (A + B) / Cos (A + B)
= (Sin A Cos B + Cos A Sin B) / (Cos A Cos B – Sin A Sin B)

Divide top and bottom by (Cos A Cos B)
Tan (A + B) = (Tan A + Tan B) / (1 – Tan A Tan B)
Put B = – C
Tan (A -C) = (Tan A – Tan C) / (1 + Tan A Tan C)

Summarising the above,
Sin (– A) = – Sin A
Cos (– A) = Cos A
Tan (– B) = Sin (– B) / Cos (– B) = – Tan B
Sin (A + B) = Sin A Cos B + Cos A Sin B (59)
Sin (A – B) = Sin A Cos B – Cos A Sin B (60)
Cos (A + B) = Cos A Cos B – Sin A Sin B (61)
Cos (A – B) = Cos A Cos B + Sin A Sin B (62)
Tan (A + B) = (Tan A + Tan B) / (1 – Tan A Tan B) (63)
Tan (A – B) = (Tan A – Tan B) / (1 + Tan A Tan B) (64)

Sin 2A, Cos 2A and Tan 2A

Put B = A;

$\text{Sin } 2A = 2 \text{ Sin } A \text{ Cos } A$ (65)

$\text{Cos } 2A = \text{Cos}^2 A - \text{Sin}^2 A$ (66)

$= 1 - 2 \text{ Sin}^2 A$ (67)

$= 2 \text{ Cos}^2 A - 1$ (68)

$\text{Tan } 2A = 2 \text{ Tan } A / (1 - \text{Tan}^2 A)$ (69)

Sin 3A and Cos 3A

Put B = 2A; $\text{Sin } 3A = 3 \text{ Sin } A - 4 \text{ Sin}^3 A$ (70)

$\text{Cos } 3A = 4 \text{ Cos}^3 A - 3 \text{ Cos } A$ (71)

Sin A Cos B etc

Add (59) and (60)

$\text{Sin } (A + B) + \text{Sin } (A - B) = 2 \text{ Sin } A \text{ Cos } B$

Therefore $\text{Sin } A \text{ Cos } B = (½) [\text{Sin } (A + B) + \text{Sin } (A - B)]$ (72)

Similarly $\text{Cos } A \text{ Cos } B = (½) [\text{Cos } (A + B) + \text{Cos } (A - B)]$ (73)

And $\text{Sin } A \text{ Sin } B = (½) [\text{Cos } (A - B) - \text{Cos } (A + B)]$ (74)

Sin A + Sin B etc

Put A + B = C and A – B = D in (72)

$\text{Sin } C + \text{Sin } D = 2 \text{ Sin } [(½)(C + D)] \text{ Cos } [(½)(C - D)]$

Writing A instead of C and B instead of D;

$\text{Sin } A + \text{Sin } B = 2 \text{ Sin } [(½)(A + B)] \text{ Cos } [½)(A - B)]$ (75)

Similarly;

$\text{Sin } A - \text{Sin } B = 2 \text{ Cos } [(½)(A + B)] \text{ Sin } [(½)(A - B)]$ (76)

And;

$\text{Cos } A + \text{Cos } B = 2 \text{ Cos } [(½)(A + B)] \text{ Cos } [(½)(A - B)]$ (77)

And;

$\text{Cos } A - \text{Cos } B = - 2 \text{ Sin } [(½)(A + B)] \text{ Sin } [(½)(A - B)]$ (78)

These results may seem pointless academic exercises, but they are used in practical applications. For example sound or vibration can usually be expressed in the form $X_{max}\text{Sin } (wt)$ so the addition of two sources of sound or vibration is analysed by (75) above.

$Sin^2 A - Sin^2 B$ etc

(75) x (76) gives;

$Sin^2 A - Sin^2 B = Sin (A + B) Sin (A - B)$ (79)

(77) x (78) gives;

$Cos^2 A - Cos^2 B = - Sin (A + B) Sin (A - B)$ (80)

$Cos^2 A - Sin^2 B = Cos^2 A - (Cos^2 A + Sin^2 A) Sin^2 B$

$= Cos^2 A (1 - Sin^2 B) - Sin^2 A Sin^2 B$

$= Cos^2 A Cos^2 B - Sin^2 A Sin^2 B$

Therefore, from (61) and (62);

$Cos^2 A - Sin^2 B = Cos (A + B) Cos (A - B)$ (81)

1 + Cos θ, 1 - Cos θ

From (68) $1 + Cos\ \theta = 2 Cos^2 [(½)\theta]$ (82)

From (67) $1 - Cos\ \theta = 2 Sin^2 [(½)\theta]$ (83)

t = Tan (θ/2)

Put t = Tan (θ/2) then $Tan\ \theta = 2t/(1 - t^2)$ (84)

$Sec^2 \theta = 1 + Tan^2 \theta = 1 + 4 t^2/(1 - t^2)^2 = (1 - 2 t^2 + t^4 + 4 t^2) / (1 - t^2)^2$

$= [(1 + t^2) / (1 - t^2)]^2$

Thus $Cos\ \theta = (1 - t^2)/(1 + t^2)$ (85)

And $Sin\ \theta = Cos\ \theta\ Tan\ \theta = 2 t/(1 + t^2)$ (86)

Properties of a triangle

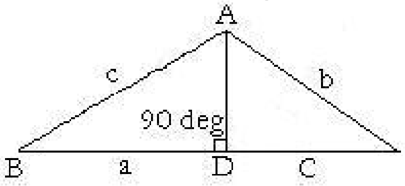


Figure 52: Properties of a triangle

Area of the triangle

= (½) x base x height, see Eqtn (39)

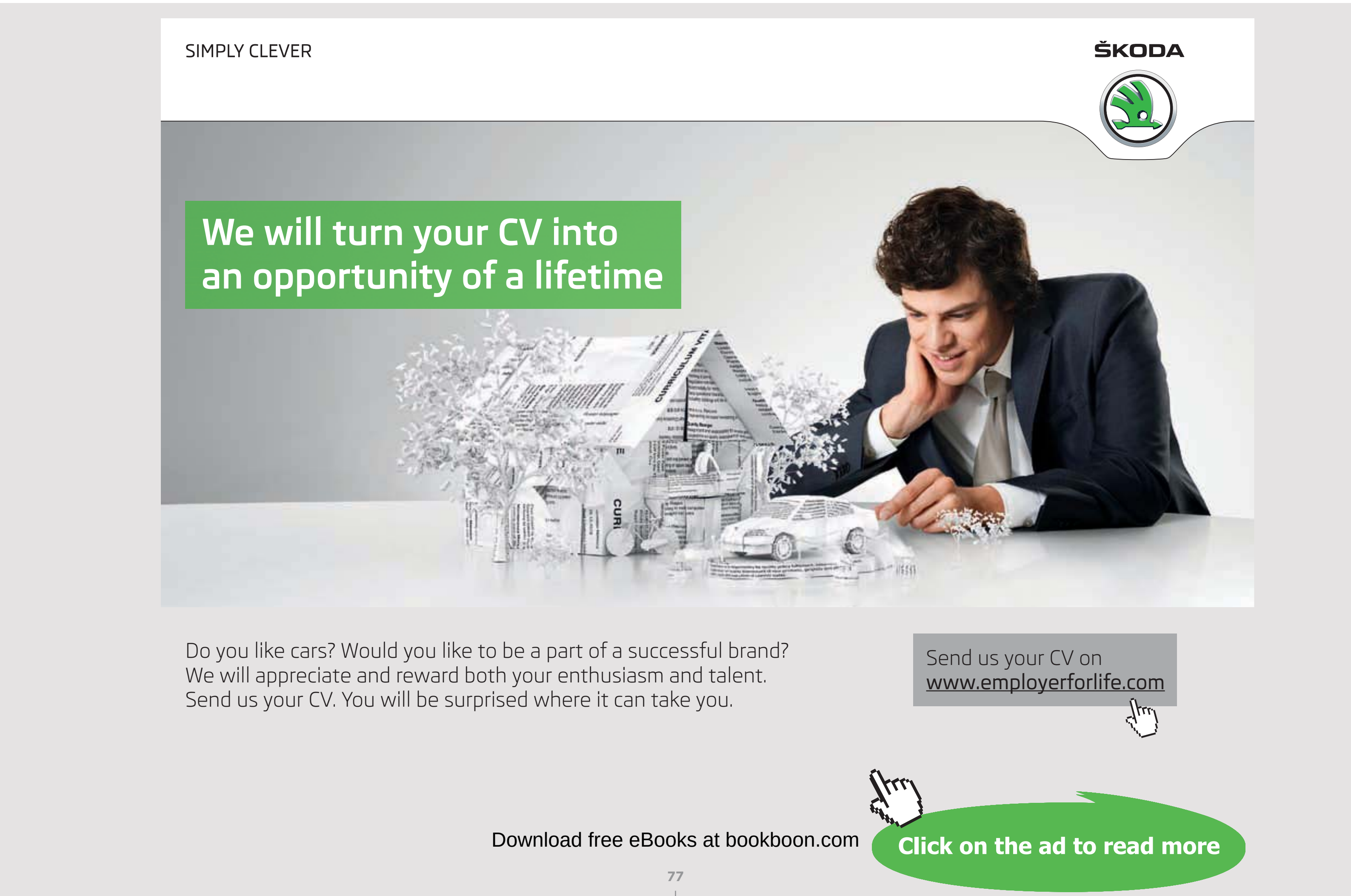
= (½) BC x AD

= (½) a x b Sin C

This is written as Area $= (1/2)\, a\, b\, \mathrm{Sin}\, C$ (87)
Similarly Area $= (1/2)\, b\, c\, \mathrm{Sin}\, A$
And Area $= (1/2)\, c\, a\, \mathrm{Sin}\, B$

Sine Formula
Equating $[(1/2)\, a\, b\, c]/\mathrm{Area}$;
$$a/\mathrm{Sin}\, A = b/\mathrm{Sin}\, B = c/\mathrm{Sin}\, C \quad (88)$$

Cosine Formula
$c^2 = AD^2 + BD^2$ from Pythagoras Eqtn (37)
$= (b\, \mathrm{Sin}\, C)^2 + (a - b\, \mathrm{Cos}\, C)^2$
$= b^2\, \mathrm{Sin}^2\, C + a^2 - 2\, a\, b\, \mathrm{Cos}\, C + b^2\, \mathrm{Cos}^2\, C$
Thus $c^2 = a^2 + b^2 - 2\, a\, b\, \mathrm{Cos}\, C$ (89)



Angles and Area in terms of the lengths of the sides

Put $s = (½)(a + b + c)$

From (89) $\text{Cos } C = (a^2 + b^2 - c^2)/(2\,a\,b)$ (90)

From (68) $\text{Cos}^2 [(½)\,C] = (1 + \text{Cos } C)/2$

$= 1/2 + (a^2 + b^2 - c^2)/(4\,a\,b)$

$= (a^2 + b^2 + 2\,a\,b - c^2)/(4\,a\,b)$

$= [(a + b)^2 - c^2]/(4\,a\,b)$

$= (a + b + c)\,(a + b - c)/(4\,a\,b)$

$= 2s\,(2s - 2c)/(4\,a\,b)$

$= s\,(s - c)/(a\,b)$

Thus $\text{Cos } [(½)\,C] = \sqrt{[s\,(s - c)/(a\,b)]}$ (91)

From (67) $\text{Sin}^2 [(½)\,C] = (1 - \text{Cos } C)/2$

$= 1/2 - (a^2 + b^2 - c^2)/(4\,a\,b)$

$= (c^2 - (a - b)^2)/(4\,a\,b)$

$= (c - a + b)\,(c + a - b)/(4\,a\,b)$

$= (s - a)\,(s - b)/(a\,b)$

Thus $\text{Sin } [(½)\,C] = \sqrt{[(s - a)\,(s - b)/(a\,b)]}$ (92)

Area of the triangle $= (½)\,a\,b\,\text{Sin } C = a\,b\,\text{Sin } [(½)\,C]\ \ \text{Cos } [(½)\,C]$

$= \sqrt{[s\,(s - a)\,(s - b)\,(s - c)]}$ (93)

5 CO-ORDINATE GEOMETRY

Cartesian Co-ordinates

A point on a graph is defined by its Cartesian Co-ordinates expressed as two numbers separated by a comma and within brackets, for example (x_1, y_1). This means starting at 0, go x_1 units in direction 0X and then y_1 units parallel to direction 0Y.

Graphical representation

Any equation relating two variables can be plotted as a curve on a graph with respect to two axes which are usually at right angles.

Any equation relating three variables can be plotted as a family of curves, choosing a value for one of the variables for each curve and plotting the other two variables.

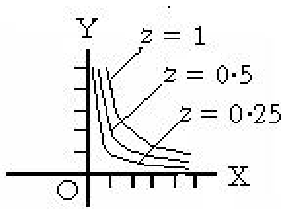


Figure 53: Graphical representation

The diagram shows a family of curves for $xyz = 1$
(for positive values of x, y and z)

Alternatively, a function with three variables can be plotted as a surface in three dimensions with axes 0X, 0Y and 0Z all mutually at right angles

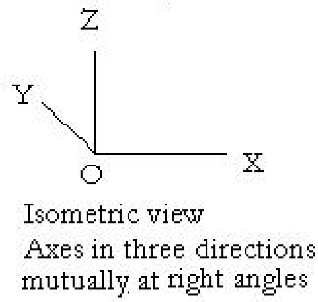


Figure 54: Plot in three dimensions

An equation with four variables can be plotted as a family of graphs, each graph depicting a family of curves.

Polar Co-ordinates

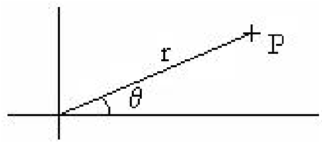


Figure 55: Polar Co-ordinates (r , θ)

A point can be defined by its Polar Co-ordinates (r , θ)
where r is the distance from the Origin and θ is the direction relative to a base line.

To convert from Polar Co-ordinates to Cartesian;

$x = \text{r Cos } \theta$ and $y = \text{r Sin } \theta$ (94)

To convert from Cartesian Co-ordinates to Polar

$\text{r} = \sqrt{(x^2 + y^2)}$ and $\theta = \text{Arc Tan } (y/x)$ (95)

(p , r) Co-ordinates

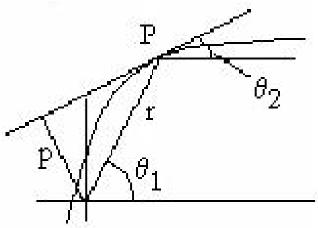


Figure 56: (p,r) Co-ordinates
The (p , r) co-ordinates for point P are shown in the diag. The tangent to the curve at point P is shown. p is the length of the perpendicular from this tangent to the origin.

Let the slope at P be m
Then $\tan \theta_2 = m$
$p/\sqrt{(r^2 - p^2)} = \tan(\theta_1 - \theta_2)$
$= (\tan \theta_1 - \tan \theta_2)/(1 + \tan \theta_1 \tan \theta_2)$
But $\tan \theta_1 = y/x$ and $\tan \theta_2 = m$

Thus $p/\sqrt{(r^2 - p^2)} = (y/x - m) / (1 + m\, y / x)$ (96)
And $r = \sqrt{(x^2 + y^2)}$ (97)

Polar Co-ordinates (r , θ) give simpler working for some problems but (p , r) co-ordinates are rarely used.

Equation for a Straight Line

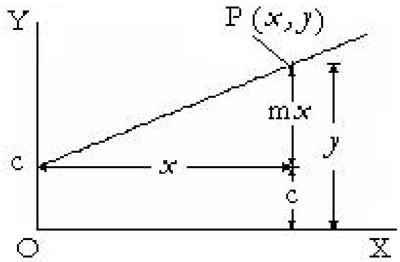


Figure 57: Straight line

(i) The general equation for a straight line

The slope m is the change in y divided by the change in x.
When $x = 0$, then $y = c$, the intercept on the OY axis.

Thus at any point on the line;
$y = mx + c$ (98)

(ii) Let the line pass through point (x_1, y_1)
therefore $y_1 = m\, x_1 + c$
Substitute for c in (98) to obtain;
$y - y_1 = m\,(x - x_1)$ (99)

This is the equation for a straight line, slope m through point (x_1, y_1)

(iii) Let the line pass through points (x_1, y_1) and (x_2, y_2)

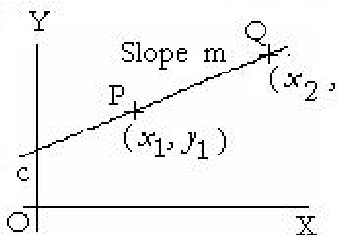


Figure 58: Line through (x_1, y_1) and (x_2, y_2)

The line, equation (99) passes through (x_2, y_2)
Therefore $y_2 - y_1 = m\,(x_2 - x_1)$
$m = (y_2 - y_1) / (x_2 - x_1)$

Substitute for m in (99) to obtain;
$(y - y_1) / (y_2 - y_1) = (x - x_1) / (x_2 - x_1)$ (100)

This is the equation for a line through points (x_1, y_1) and (x_2, y_2)

Distance between two points

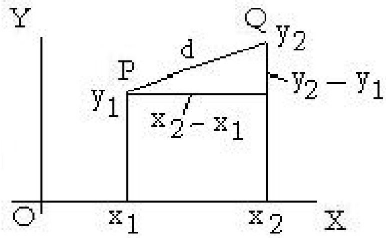


Figure 59: Distance between two points
Let d be the distance between points P at (x_1, y_1) and Q at (x_2, y_2);

$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$ (101)

Distance of point (x_1 , y_1) from line y = mx + c

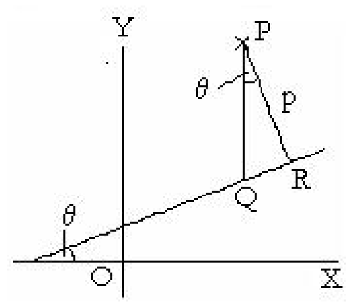


Figure 60: Distance of (x_1 , y_1) from y = mx + c

P is point (x_1 , y_1)
Q is point (x_1 , m x_1 + c)

PQ = y_1 – (m x_1 + c)
p = PR = PQ Cos θ
= PQ / Sec θ
= PQ /√ (1 + $\tan^2$ θ)
= PQ /√ (1 + m^2)

$$p = (y_1 - mx_1 - c) / \sqrt{(1 + m^2)} \qquad (102)$$

Angle between two lines

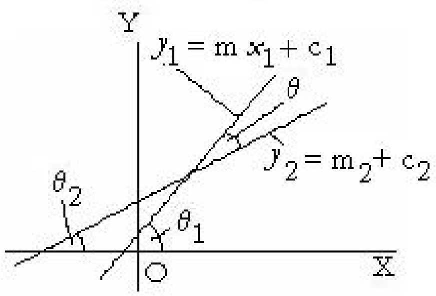


Figure 61: Angle between two lines

$\theta = \theta_1 - \theta_2$

$$\tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$\tan\theta = (m_1 - m_2)/(1 + m_1m_2)$ (103)

The lines cross orthogonally if $m_1m_2 = -1$ (104)

Two straight lines

Two straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$ can be represented by a single equation;

$(y - m_1x - c_1)\,(y - m_2x - c_2) = 0$ (105)

Conic Sections

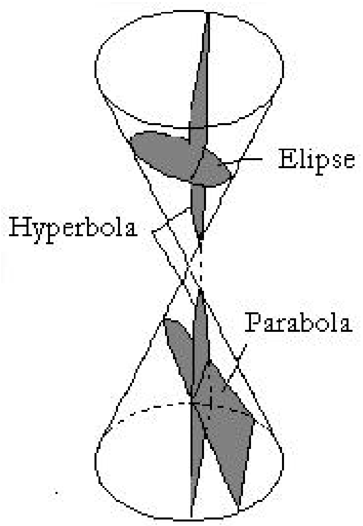


Figure 62: Conic Sections

Conic Sections are the outline shape of a section through a cone.

If the cone is sliced perpendicular to its axis, the shape of the section is a circle.
If the cone is sliced at an angle to the circle, the shape is an ellipse.
If the cone is sliced parallel to the edge of the cone, the shape is a parabola.
If the cone is sliced at an angle closer to the axis of the cone, the shape is a hyperbola.

Circle

(i) Circle with the centre at O

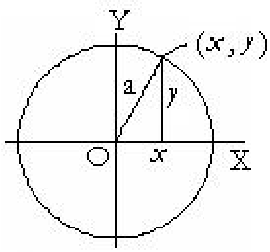


Figure 63: Circle with centre at O

At any point on the circumference;

$x^2 + y^2 = a^2$ (106)

(ii) Circle centre at (g , h)

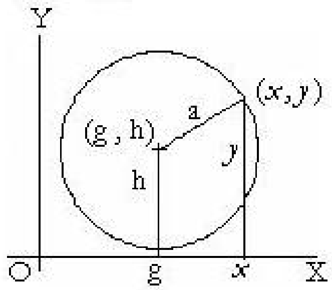


Figure 64: Circle centre at (g,h)

At any point on the circumference;

$(x - g)^2 + (y - h)^2 = a^2$ (107)

Or;

$x^2 + y^2 - 2gx - 2hy + k = 0$ (108)

where $k = g^2 + h^2 - a^2$

Thus an equation is a circle if;

x^2 and y^2 have the same coefficient
and there is no xy term
and there are no powers higher than 2

Ellipse

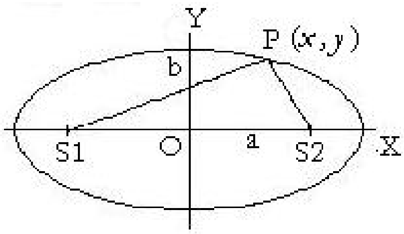


Figure 65: Ellipse

An ellipse is a squashed circle.

In the diagram, the value of y is the value for a circle reduced by factor of (b/a).
Thus $y^2 = (b/a)^2 (a^2 - x^2)$

Hence $x^2/a^2 + y^2/b^2 = 1$ (109)

Foci are at the points;
S_1 is at $(-\sqrt{(a^2 - b^2)}\ ,\ 0)$
S_2 is at $(\sqrt{(a^2 - b^2)}\ ,\ 0)$ (110)

$(S_2P)^2 = y^2 + [\sqrt{(a^2 - b^2)} - x]^2$
$= b^2 (1 - x^2 / a^2) + a^2 - b^2 - 2x\sqrt{(a^2 - b^2)} + x^2$
$= - b^2x^2/a^2 + a^2 + x^2 - 2x\sqrt{(a^2 - b^2)}$
$= (x^2/a^2)(a^2 - b^2) - 2(x a/a)\sqrt{(a^2 - b^2)} + a^2$
$= [a - (x/a)\sqrt{(a^2 - b^2)}]^2$
$S_2P = a - [x\sqrt{(a^2 - b^2)}]/a$

Similarly
$S_1P = a + [x\sqrt{(a^2 - b^2)}]/a$

$S_1P + S_2P = 2a$ (111)

Eccentricity of an Ellipse

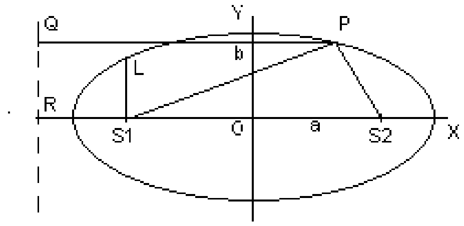


Figure 66: Eccentricity of an ellipse

By definition, eccentricity
$e = S_1P / PQ$

A line QR can be found where the value of e is the same for all points on an ellipse.

From above;
$S_1P = a + [x\sqrt{(a^2 - b^2)}]/a$
$PQ = RO + x$
$e\,RO + e\,x = S_1P = a + [x\sqrt{(a^2 - b^2)}]/a$
This is true for all values of x, therefore coefficients can be equated
Coefficient of x
$e = [\sqrt{(a^2 - b^2)}]/a = \sqrt{(1 - b^2/a^2)}$ (112)
Constant term $e\,RO = a$
Therefore $RO = a/e = a^2/\sqrt{(a^2 - b^2)}$ (113)

Also S_1L is called the Semi Latus Rectum and its length is given by;
$S_1L = b\sqrt{(1 - x^2/a^2)} = (b/a)\sqrt{[a^2 - (a^2 - b^2)]} = b^2/a$ (114)

Parabola

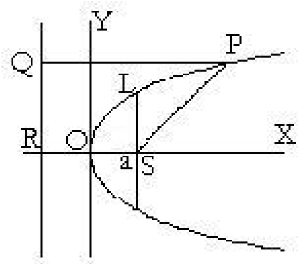


Figure 67: Parabola

The equation of a Parabola, vertex at O and axis on OX

$y^2 = 4\,a\,x$ (115)

Focus S is at point (a , 0) (116)

Semi Latus Rectum SL = 2a (117)

Eccentricity of a Parabola

Let the parabola be $y^2 = 4\,a\,x$

SP = e PQ

$SP^2 = (x - a)^2 + y^2 = x^2 - 2ax + a^2 + 4ax = (x + a)^2$

$SP = x + a$ (118)

$PQ = RO + x$

Therefore $x + a = e\,RO + e\,x$

Equating coefficients;

$e = 1$ (119)

$RO = a$ (120)

Hyperbola, orthogonal asymptotes

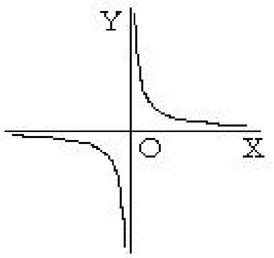


Figure 68: Hyperbola orthogonal asymptotes

The equation of a hyperbola with orthogonal asymptotes along the OX and OY axes;

$x\,y = c^2$ (121)

Hyperbola, asymptotes not orthogonal

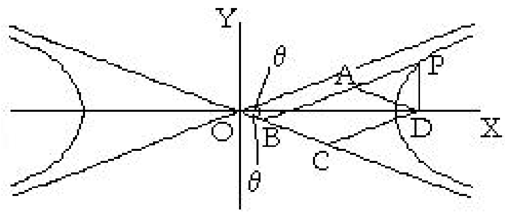


Figure 69: Hyperbola with asymptotes not orthogonal

Point P (x, y) is on the hyperbola
OD = x and DP = y
Since a hyperbola, OB. BP = c^2

CD and BA are parallel to an asymptote
AD and BC are parallel to the other asymptote
OCD is an isosceles triangle, therefore OC = CD = x / (2 Cos θ)
Also CD = AB

Similarly

$BC = AD = AP = y / (2 \sin\theta)$

$OB = OC - BC = x / (2 \cos\theta) - y / (2 \sin\theta)$

$BP = AB + AP = x / (2 \cos\theta) + y / (2 \sin\theta)$

$c^2 = OB \,.\, BP = [x / (2 \cos\theta)]^2 - [y / (2 \sin\theta)]^2$

$[x / (2 c \cos\theta)]^2 - [y / (2 c \sin\theta)]^2 = 1$

Put $a = 2 c \cos\theta$ and $b = 2 c \sin\theta$

Thus the general equation for a hyperbola with asymptotes that are not orthogonal is;

$$(x / a)^2 - (y / b)^2 = 1 \qquad (122)$$

Eccentricity of a hyperbola

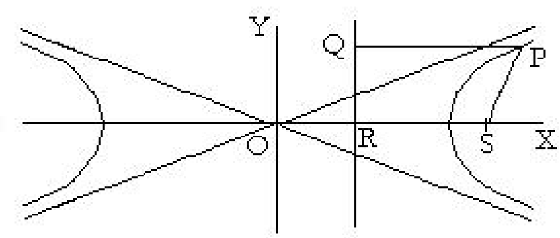


Figure 70: Eccentricity of a hyperbola

P is point (x, y)

$SP = e\ QP$

Let $OS = \alpha$ and $OR = \beta$

Then $SP^2 = y^2 + (x - \alpha)^2$

and $QP = x - \beta$

At P $x^2/a^2 - y^2/b^2 = 1$

Therefore $y^2 = b^2 [(x/a)^2 - 1]$

$SP^2 = x^2 b^2 / a^2 - b^2 + x^2 - 2 \alpha x + \alpha^2$

$\quad = x^2 (a^2 + b^2) / a^2 - 2 \alpha x + \alpha^2 - b^2$

But $SP^2 = e^2 QP^2 = e^2 x^2 - 2 e^2 \beta x + e^2 \beta^2$

Equating coefficients;

$$e^2 = (a^2 + b^2) / a^2 \qquad e = \sqrt{(1 + b^2 / a^2)} \qquad (123)$$

Also $\alpha = e^2 \beta$ and $\alpha^2 - b^2 = e^2 \beta^2$

Eliminating β and substituting for e

$\alpha^2 - b^2 = (\alpha/e)^2 = \alpha^2 a^2/(a^2 + b^2)$

$\alpha^2(a^2 + b^2) - b^2 (a^2 + b^2) = \alpha^2 a^2$ therefore $\alpha = \sqrt{(a^2 + b^2)}$

Therefore Foci are at points
$(\sqrt{(a^2 + b^2)}, 0)$ and $(-\sqrt{(a^2 + b^2)}, 0)$ (124)

And $OR = \beta = \alpha / e^2 = \sqrt{(a^2 + b^2)}\, a^2 / (a^2 + b^2)$
Therefore $OR = a^2 / \sqrt{(a^2 + b^2)}$ (125)

Semi Latus Rectum L is the value of y when x = value for Focus
$x^2 / a^2 - L^2 / b^2 = 1$ where $x^2 = a^2 + b^2$
$L^2 = b^2 [(a^2 + b^2) / a^2 - 1) = b^2 [(a^2 + b^2 - a^2) / a^2] = b^4 / a^2$
Therefore $L = b^2 / a$ (126)

Polar Equation for a Conic Section

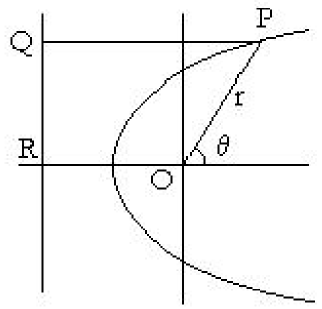


Figure 71: Polar equation for a conic section

Take the Origin at a Focus

Then $OP = e\, QP$
$r = e (OR + r \cos\theta)$
When $\theta = \pi/2$, $r = L$ the Semi Latus Rectum
Therefore $L = e\, OR$
$r = L + e\, r \cos\theta$

Thus the general equation for a Conic Section is;
$L / r = 1 - e \cos\theta$ (127)
where L is the Semi Latus Rectum
If $e < 1$, the curve is an ellipse
If $e = 1$, the curve is a parabola
If $e > 1$, the curve is a hyperbola

6 LOGORITHMS

Definition of Logarithms

If $a^x = m$ then by definition $\log_a m = x$ (128)

Let $\log_a m = x$ and $\log_a n = y$

Then $a^x = m$ and $a^y = n$

Therefore $m\,n = a^x a^y = a^{(x+y)}$

Therefore $\log_a mn = x + y = \log_a m + \log_a n$

$$\log_a m + \log_a n = \log_a (mn) \quad (129)$$

Similarly

$$\log_a m - \log_a n = \log_a (m / n) \quad (130)$$

Also $n \log_a m = \log_a m^n$ (131)

Change the Base

Let $\log_a m = x$ $\quad \log_a b = y$ $\quad$ and $\log_b m = z$
Then $a^x = m$ $\quad a^y = b$ $\quad$ and $b^z = m$
Therefore $\quad a^x = m = b^z = (a^y)^z = a^{yz}$
Therefore $\quad x = yz$ $\quad$ or $z = x / y$

Therefore $\log_b m = \log_a m / \log_a b$ (132)

Plotting Logarithmic Functions

(i) Consider the equation $y = a\,x^n + b$ (133)

Then $\quad \log (y - b) = n \log x + \log a$

Put $\quad Y = \log (y - b)$
and $\quad X = \log x$

Substituting for Y and X
$$Y = n X + \log a$$
This is the equation of a straight line with slope n.
Thus if x and y are experimental results and a relation $y = a\,x^n + b$ is suspected, then b is the value of y when $x = 0$.

$\log (y - b)$ can then be plotted against $\log x$. If the result is a straight line, then the relation is confirmed and n can be measured as the slope of the line.

Graph Paper with log scales on the OX and OY axes are available, or alternatively the plotting can be done directly by computer.

(ii) Consider the equation $y = p\,a^x + q$ (134)

Then $\quad \log(y - q) = x \log a + \log p$
q is the asymptotic value of y as x approaches minus infinity

Plot $\log (y - q)$ against x and if the result is a straight line then the relation is confirmed.
The slope of the line is $\log a$, and the intercept on the OY axis is $\log p$
Graph Paper with a log scale on the OY axis and a linear scale on the OX axis is available or alternatively the plotting can be done by computer.

7 PERMUTATIONS AND COMBINATIONS

nPr

No of ways of filling ' r ' different spaces by selecting from ' n ' different items is;
1st choice = n possibilities
2nd choice = n − 1 possibilities
3rd choice = n − 2 possibilities
r th choice = n − r + 1 possibilities

Therefore the total number of ways;
nPr = n (n − 1) (n − 2) ....(n − r + 1)

Therefore nPr = n ! / (n − r) ! (135)
nPr is known as number of Permutations of n things, r at a time.

nCr

Number of ways of filling ' r ' similar spaces by selecting from ' n ' different items.

Number of ways;

$$nCr = \frac{\text{Number of ways of filling } r \text{ different spaces from } n \text{ items}}{\text{Number of ways of filling } r \text{ different spaces from } r \text{ items}}$$

Therefore nCr = n ! / [(n − r) ! r !) (136)
nCr is known as number of Combinations of n things, r at a time

Examples

Example 1
Suppose that there are 5 ways of going from A to B, and 3 ways of going from B to C
Number of ways of going from A to C via B is 5 times 3 = 15

Example 2
Find number of different arrangements of all four letters a, b, c and d
4 ways to fill the 1st space
3 2nd
2 3rd
1 4th
Number of ways is 4 x 3 x 2 x 1 = 4 !

Example 3
A girl has 5 hats. How many ways can she wear them if she wears one on each day of the week for seven days.
Number of ways = $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$

Example 4
Number of ways of arranging n unlike objects in a row is $n!$

Example 5
Number of ways of arranging n unlike objects in a circle regarding clockwise and anti-clockwise arrangements as different is $(n-1)!$
ie fix one object and arrange the remainder.

Example 6
Number of ways of arranging n objects in a circle if no distinction is made between clockwise and anti-clockwise.
Number of ways = 1/2 x Example5 = $[(n-1)!]/2$

Example 7
Number of ways of making a three lettered word with 26 blocks each marked with a different letter.

Number of ways of filling the 1st space		= 26
Number of ways	2nd	25
Number of ways	3rd	24

Number of ways = 26 x 25 x 24 = 26 ! / (26 − 3) !
This a case of nPr

Example 8
How many numbers greater than 7000 can be made from the digits 3, 5, 7, 8 and 9. if no digit is repeated.
If the number contains 5 digits, it can be formed in 5P5 = 5 ! ways
If the number contains 4 digits, the first digit can be 7, 8 or 9 ie 3 ways
Whichever left hand digit is chosen, the arrangement can be completed in 4P3 ways
Number of ways = 5P5 + 3 x 4P3
= 5 x 4 x 3 x 2 x 1 + 3 x (4 x 3 x 2) = 120 + 72 = 192

Example 9
Number of ways or arranging n things in a row when there are p alike of one kind, q alike of another kind r alike of another kind etc.

Number of ways of arranging n unlike things in a row = n !
In any one of these ways, the p like things can be arranged amongst themselves p ! ways
Number of ways of arranging n things of which p are alike = n!/p!

If there are p alike things and q alike things,
the number of ways = (n!/p!) / q! = n ! / (p ! q !)

Hence number of ways of arranging n things in a row when there are p alike of one kind, q alike of another kind, r alike of another kind etc = n! / (p! q! r! ......)

Example 10
Find the number of ways that a basket can be filled with r objects selected from a total of n objects. No regard is to be paid to the order in which the objects are selected.

If regard is paid to the order of selection,
then number of ways = nPr

For any one of these ways, there are r! ways of placing the r objects in order.
Thus the required number of ways is nCr = nPr / r! = n! / [(n − r)! r!]

Example 11
Find the number of ways of dividing $(p + q + r)$ unlike things into 3 unequal groups containing respectively p, q and r things.

A group of p things can be selected in $(p + q + r)Cp$ ways
From the remaining $q + r$ things, a group of q things can be selected in $(q + r)Cq$ ways
This leaves r things for the third group

Number of ways $= (p + q + r)Cp \times (q + r)Cq$
$= (p + q + r)! / [\, p! \,(q + r)!] \times (q + r)! \,(\, q! \, r! \,)$
$= (p + q + r)! / (p! \, q! \, r!)$

Example 12
Find the number of selections from n unlike things taking any number at a time

Each thing may be selected or rejected, ie it may be disposed of in two ways
Number of ways of disposing of n things $= 2 \times 2 \times 2 \times \ldots\ldots\ldots$ to n factors $= 2^n$
This includes the one case where all are rejected.

Number of ways if at least one is selected $= 2^n - 1$

Example 13
Given k unlike things plus p alike things of one kind plus q alike things of another kind plus r alike things of another kind etc, find the number of selections taking any number at a time.

From the p things, we can select $0, 1, 2, \ldots\ldots$ or p
We can dispose of the p things in $(p + 1)$ ways
Similarly we can dispose of the q things in $(q + 1)$ ways
And we can dispose of the r things in $(r + 1)$ ways

In addition the k things can be disposed of as in example 12

Total number of ways $= 2^k (p + 1)(q + 1)(r + 1)$ including the one case when all are rejected.

Number of ways if at least one thing is chosen is $2^k (p + 1)(q + 1)(r + 1) - 1$

Binominal Theorem

$(x + a)^n = (x + a)\ (x + a)\ (x + a)..............(x + a)$ n factors

Coefficient of $a^r x^{n-r}$ = number of ways of selecting " (a^r)" out of n factors
= nCr

Therefore;

$$(x + a)^n = x^n + n\,a\,x^{n-1} + [n\,(n-1) / 2!]\,a^2 x^{n-2} + [n\,(n-1)\,(n-2) / 3!]\,a^3 x^{n-3} \,.....\, + \,......$$
$$+ \,n! / [(n-r)!\,r!]\,a^r x^{n-r} + \,........... + a^n \qquad (137)$$

Example

$$(x + a)^4 = x^4 + 4\,a\,x^3 + 4.3/2\,a^2 x^2 + 4.3.2/(2.3)\,a\,x^3 + x^4$$
$$= x^4 + 4\,a\,x^3 + 6\,a^2 x^2 + 4\,a^3 x + a^4$$

The theorem is found to be still true if n is not an integer or if n is negative,
provided $(-1 < x < +1)$

Example

$(1 + x)^{-1} = 1 + (-1)\, x + (-1)(-2) / 2\, x^2 + (-1)(-2)(-3) / (2.3)\, x^3 + ...$

$= 1 - x + x^2 - x^3 + x^4 .....$ to infinity

This is only true if $-1 < x < +1$

Put $x = -0.1$

$(1 + x)^{-1} = (1 - 0.1)^{-1} = 1/(0.9) = 10/9 = 1.11111$ recurring

But $1 - x + x^2 - x^3 + x^4 .....$ to infinity $= 1 + 0.1 + 0.01 + 0.001 + 0.0001 + ...$

$= 1.1111$ recurring

Thus it is true when $x = -0.1$

A computer program can show that the binominal theorem gives the right answer for values of n that are not positive integers provided $-1 < x < +1$;

```
PRINT "Evaluate binominal (1 + x)^n"
INPUT "Input x ", x#
INPUT "Input index n ", N
INPUT "Input number of terms ", M
S1# = (1 + x#) ^ N
S2# = 1
term# = 1
FOR r = 1 TO M
term# = term# * x# * (N + 1 - r) / r
S2# = S2# + term#
NEXT r
PRINT "(1 + x)^n"; TAB(25); S1#
PRINT "Binominal (1 + x)^n  "; TAB(25); S2#; ""
```

This program runs on Qbasic. On Visual basic, input should be via text boxes and space should be reserved clear of Command Buttons and Text Boxes for the display.

8 MATRICES AND DETERMINANTS

Matrices

A Matrix is a convenient way to record numerical data.

$$\mathbf{A} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$ is a Matrix with 4 rows and 3 columns

A Matrix has Dimensions (rows x columns), rows before columns.
The above Matrix **A** has Dimensions (4 x 3) or **A** is of order 4 x 3

A Row Vector is a Matrix with one row
A Column Vector is a Matrix with one column
A Square Matrix has the same number of rows and columns

A Diagonal Matrix is a Square Matrix with all elements zero except on a diagonal eg

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$$

The sum of the elements of a Diagonal Matrix is the Trace

A Scalar Matrix is a Diagonal Matrix with all the elements on the diagonal the same

eg $$\begin{vmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{vmatrix}$$

A Unit Matrix or an Identity Matrix is a Scalar Matrix with k. = 1

A Symmetrical Matrix is a Square Matrix with elements a mirror image across the diagonal

eg $$\begin{vmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$ is the Transpose of $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

ie the row and column of each element are swapped
The Transpose of a Matrix **A** is written as **A'**

Matrix Data

For example, two customers A and B, purchase items P, Q and R.
Their number of purchases over two weeks are recorded as;

Week 1	Item P	Item Q	Item R	Expressed as a Matrix
Customer A	2	1	1	$\lvert\ 2 \quad 1 \quad 1\ \rvert$
Customer B	1	0	2	$\lvert\ 1 \quad 0 \quad 2\ \rvert$

Week 2	Item P	Item Q	Item R	Expressed as a Matrix
Customer A	3	0	2	$\lvert\ 3 \quad 0 \quad 2\ \rvert$
Customer B	2	2	1	$\lvert\ 2 \quad 2 \quad 1\ \rvert$

Their total for the two days in Matrix form is;

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 1 & 3 \\ 3 & 2 & 3 \end{vmatrix}$$

Two Matrices with the same Dimensions can be **Added** or **Subtracted**
Each element in one Matrix is added to (or subtracted from) the corresponding element in the other Matrix.

Suppose Item P costs £2, Item Q costs £1 and Item R costs £1.5
The cost for the two days is given by the Matrices;

$$\left|\begin{matrix} 5 & 1 & 3 \\ 3 & 2 & 3 \end{matrix}\right| \text{ x } \left|\begin{matrix} 2 \\ 1 \\ 1.5 \end{matrix}\right| = \left|\begin{matrix} 5\text{x}2+1\text{x}1+3\text{x}1.5 \\ 3\text{x}2+2\text{x}1+3\text{x}1.5 \end{matrix}\right| = \left|\begin{matrix} 15.5 \\ 12.5 \end{matrix}\right|$$

This is a Matrix **Multiplication**

If next week the customers buy the same every day as given by the Matrix $\left|\begin{matrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}\right|$

Then for 5 days, their total purchases are $\left|\begin{matrix} 10 & 0 & 5 \\ 5 & 5 & 5 \end{matrix}\right|$

Thus 5 times the Matrix is a Matrix with every element multiplied by 5

Thus a Matrix can be **Multiplied by a Scalar**
Every element in the Matrix is multiplied by the Scalar

If they pay full price on the Saturday but there is a Sale with 50% discount on the Sunday the Matrices for Saturday and Sunday become;

$$\left|\begin{matrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}\right| \text{ x } \left|\begin{matrix} 2 & 1 \\ 1 & 0.5 \\ 1.5 & 0.75 \end{matrix}\right| = \left|\begin{matrix} 5.5 & 2.75 \\ 4.5 & 2.25 \end{matrix}\right|$$

The values in the final Matrix are obtained by 2x2+0x1+1x1.5 = 5.5
1x2+1x1+1x1.5 = 4.5, 2x1+0x0.5+1x0.75 = 2.75 and 1x1+1x0.5+1x0.75 = 2.25

Thus Matrices can be multiplied provided that the number of Columns in the first Matrix is the same as the number of Rows in the second. The result has the same number of Rows as the first and the same number of Columns as the second Matrix

$$\left|\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right| \text{ X } \left|\begin{matrix} g & j \\ h & k \\ i & l \end{matrix}\right| = \left|\begin{matrix} (ag+bh+ci) & (aj+bk+cl) \\ (dg+eh+fi) & (dj+ek+fl) \end{matrix}\right|$$

Note that, if **A** and **B** are Matrices, then **A** x **B** does not in general equal **B** x **A**
The order cannot be changed

Simultaneous Equations expressed as a Matrix

Let x, y and z be defined by the linear simultaneous equations;

Equation 1 $\quad a_1 x + b_1 y + c_1 z = d_1$
Equation 2 $\quad a_2 x + b_2 y + c_2 z = d_2$
Equation 3 $\quad a_3 x + b_3 y + c_3 z = d_3$
Using the rules for multiplication of Matrices, this set of equations can be written in Matrix form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix} \quad \text{or more shortly} \quad \begin{vmatrix} a_1 & b_1 & c_1 & : d_1 \\ a_2 & b_2 & c_2 & : d_2 \\ a_3 & b_3 & c_3 & : d_3 \end{vmatrix}$$

The equations can be solved while in this shorter Matrix form.
Each line in the shorter Matrix defines one equation. Line 1 can be replaced by the sum or difference of line 1 and any other line or lines. The result can then be put in the Matrix to replace line 1. The operation can be repeated again and again till a diagonal Matrix is obtained.

Example. Solve the simultaneous equations

$$\begin{vmatrix} 3 & -5 & 6 : 12 \\ 2 & 8 & -7 : 10 \\ 4 & -3 & 5 : \ 8 \end{vmatrix}$$

Replace line 1 by 8 x line 1 plus 5 x line 2, ie.	$\lvert\ 34 \quad 0 \quad 13 : \quad 146\ \rvert$
Replace line 3 by 3 x line 1 minus 5 x line 3, ie	$\lvert -11 \quad 0 \quad -7 : \quad -4\ \rvert$
Replace line 1 by 7 x new 1 plus 13 x new 3, ie	$\lvert\ 95 \quad 0 \quad 0 : \quad 970\ \rvert$
Replace line 2 by 2 x line 1 minus 3 x line 2, ie	$\lvert\ 0 \quad -34 \quad 33 : \quad -6\ \rvert$
Replace line 3 by line 3 minus 2 x line 2, ie	$\lvert\ 0 \quad -19 \quad 19 \quad -12\ \rvert$
Replace line 2 by 19 x new 2 minus 33 x new 3	$\lvert\ 0 \quad -19 \quad 0 : \quad 282\ \rvert$
Similarly obtain a new line 3	$\lvert\ 0 \quad 0 \quad -95 : 1470\ \rvert$

Thus the Matrix for the simultaneous equations becomes

$$\begin{vmatrix} 95 & 0 & 0 : & 970 \\ 0 & -19 & 0 : & 282 \\ 0 & 0 & -95 : & 1470 \end{vmatrix}$$

Multiply out to obtain **;**
$x = 970/95$, $y = -282/19$ and $z = -1470/95$

Determinants

Determinants are a shorthand way of showing some expressions that are in regular use;

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_1 \\ a_3 & b_3 & c_1 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

Each term contains one element from each row and one element from each column, ie every combination of terms containing abc and 123

With the letters in the sequence abc, terms with the numbers in the sequence 123123 are positive, others are negative

9 SERIES

Summation of Finite Series

$S_n = U_1 + U_2 + U_3 + ... + U_r + ... + U_n$

ie $S_n = \Sigma_{r=1 \text{ to } n} U_r$

Difference Method

Suppose some function of r can be found such that;

$U_r = F(r + 1) - F(r)$

Then $S_n = F(n + 1) - F(1)$ (138)

Induction Method

Can only be used to prove a stated result

To prove $\Sigma_{r=1 \text{ to } n} U_r = f(n)$

(i) Assume the formula is true for a particular value of n, say $n = k$

(ii) Then prove that , if this is so, then the formula is also true for $n = k + 1$

(iii) Prove that the formula is true for $n = 1$

therefore the formula is true for $n = 2$

Also true for $n = 3$ etc

Arithmetical Progression (or AP)

$S_n = a + (a + d) + (a + 2d) + ... + [a + (r - 1)d] + ... + [a + (n - 1)d]$

Add first term to last term, add second term to second to last term etc

$[a + a + (n - 1)d] + [a + d + a + (n - 2)d] + ...$ for n terms

$= n$ terms each $[2a + (n - 1)d]$

Therefore $2S_n = n (2a + (n - 1)d)$

And $S_n = (½) n [2a + (n - 1) d]$ (139)

Example.

Show that the sum of the first n odd numbers is a perfect square

This an AP with $a = 1$ and $d = 2$

The Sum $= (½) n [2 + (n - 1) 2] = (½) n [2 + 2n - 2] = (½) n [2 n] = n^2$

Geometrical Progression (or GP)

$S_n = a + ap + ap^2 + ... + ap^{r-1} + ... + ap^{n-1}$
Consider $(1 - p) S_n = a + ap + ap^2 + ... + ap^{r-1} + ... + ap^{n-1}$
$\qquad - ap - ap^2 - ..... - ap^{r-1} - ..... - ap^{n-1} - ap^n$
Therefore $S_n = a (1 - p^n) / (1 - p)$ (140)

Sum of first n numbers, squares and cubes

$S_1 = 1 + 2 + 3 + ... + n$
This is an AP, therefore $S_1 = n (n + 1) / 2$ (141)

$S_2 = 1^2 + 2^2 + 3^2 + ... + n^2$
Consider $\Sigma_{r=1 \text{ to } n} [(r + 1)^3 - r^3]$
$= \Sigma_{r=1 \text{ to } n} [r^3 + 3 r^2 + 3 r + 1 - r^3]$
$= \Sigma_{r=1 \text{ to } n} [3 r^2 + 3 r + 1]$

Therefore $3S_2 + 3S_1 + n = (n + 1)^3 - 1^3$
$6S_2 + 3n (n + 1) + 2n = 2n^3 + 6n^2 + 6n$
$6S_2 = 2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n = 2n^3 + 3n^2 + n$
Hence $S_2 = (1/6) n (n + 1) (2n + 1)$ (142)

$S_3 = 1^3 + 2^3 + 3^3 + ... + n^3$
Consider $f(r) = [(r - 1) r)]^2$
$f(r + 1) - f(r) = [r (r + 1)]^2 - [(r - 1) r)]^2$
$= r^2 [(r^2 + 2 r + 1) - (r^2 - 2 r + 1)]$
$= r^2 [4 r] = 4r^3 = 4U_r$
$4S_3 = f(n + 1) - f(1) = [(n + 1) n]^2 - 0$
$S_3 = [(n + 1) n / 2]^2$ (143)

Examples on finite series

Example 1
$S_n = \Sigma_{r=1 \text{ to } n} r(r + 1)(r + 2)$

Consider $f(r) = (r -1)r(r + 1)(r + 2)$
Then $f(r + 1) = r(r + 1)(r + 2)(r + 3)$
$f(r + 1) - f(r) = r(r + 1)(r + 2)(r + 3) - (r - 1)r(r + 1)(r + 2)$
$= r(r + 1)(r + 2)(r + 3 - r + 1)$
$= 4U_r$
Therefore $S_n = (1/4) [f(n + 1) - F(1)]$
$= (1/4) n (n + 1) (n + 2) (n + 3)$

The method may be used for all series of the type;

$$U_r = (a + rd)\,[a + (r + 1)\,d]$$

or

$$U_r = (a + rd)\,[a + (r + 1)\,d]\,[a + (r + 2)\,d] \quad \text{etc}$$

Example 2

Prove $\Sigma_{r=1 \text{ to } n}\, r^3 = [n\,(n + 1) / 2]^2$ ie result (143)

Suppose it is true for $n = k$

$$\Sigma_{r=1 \text{ to } k}\, r^3 = [k\,(k + 1) / 2]^2$$

But

$$\begin{aligned} \Sigma_{r=1 \text{ to } k+1}\, r^3 &= \Sigma_{r=1 \text{ to } k}\, r^3 + U_{k+1} \\ &= (1/4)\, k^2\,(k + 1)^2 + (k + 1)^3 \\ &= (1/4)\,(k + 1)^2\,(k^2 + 4k + 4) \\ &= (1/4)\,(k + 1)^2\,(k + 2)^2 \end{aligned}$$

Therefore, if the formula is true for $n = k$, then it is true for $n = k+1$
But the formula is true for $n=1$ because $1^3 = (1/4)\, 1^2\, 2^2$
Therefore the formula is true for all positive integral values of n

Example 3

$$S_n = \Sigma_{r=1 \text{ to } n} [a + (r - 1)d]\, x^{r-1}$$

$$S_n = a + (a + d)\, x + (a + 2d)\, x^2 + \ldots + [a + (n - 1)d]\, x^{n-1}$$

Therefore $x S_n = a x + (a + d)\, x^2 + \ldots.. + [a + (n - 1)d]\, x^n$

$$S_n (1 - x) = a + d\,x + d\,x_2 + \ldots\ldots.. + d\,x^{n-1} - [a + (n - 1)d]\, x^n$$

Using the result for a GP;

$$S_n (1 - x) = a + d\,x (1 - x)^{n-1} / (1 - x) - [a + (n - 1)d]\, x^n$$

Therefore $S_n = [a - \{a + (n - 1)\, d\}x^n] / (1 - x) + d\,x (1 - x^{n-1}) / (1 - x)^2$

Infinite Series

Let S = Limit as n tends to infinity $[\Sigma_{r=1 \text{ to } n} U_r]$

(i) Series is Convergent if S is finite

(ii) Series is Divergent if S tends to plus or minus infinity

(iii) Series is Oscillating if S oscillates

Example $S_n = 1 - 1 + 1 - 1 + 1 - \ldots.$ Limit as n tends to infinity S_n oscillates finitely

$S_n = 1 - 2 + 3 - 4 + 5 - 6 + \ldots$ Limit as n tends to infinity S_n oscillates infinitely

General Properties of Limits

If $\Sigma\, U_r$ is a Convergent series, then Limit as r tends to infinity $U_r = 0$

If Limit as r tends to infinity $U_r = 0$, the Series may be Convergent or Divergent

For Example

The Sum of the Series $\Sigma\, (1/r)$ tends to infinity as n tends to infinity

Sum to infinity of a GP

The Sum to infinity of a GP is finite if $p < 1$

Therefore S to infinity $= a (1 - p^{\infty}) / (1 - p) = a / (1 - p)$

Tests for Convergence if all terms positive

(i) Comparison Tests

a) Let ΣV_n be a Convergent Series with all terms positive
and ΣU_n be another Series with all terms positive
Then ΣU_n is also Convergent if;
Limit as n tends to infinity $[U_n / V_n] =$ a positive constant

b) If ΣV_n is a Divergent Series but other conditions of (a) are met, then ΣU_n is also Divergent

(ii) Ratio Test

a) Let ΣU_n be a Series with all terms positive
ΣU_n is Convergent, if Limit as n tends to infinity $[U_n / U_{n+1}] > 1$

b) ΣU_n is Divergent, if Limit as n tends to infinity $[U_n / U_{n+1}] < 1$

c) If Limit as n tends to infinity $[U_n / U_{n+1}] = 1$, Series may be Convergent or Divergent

d) If Limit as n tends to infinity $[U_n / U_{n+1}]$ Oscillates above and below 1, no conclusion can be drawn

Absolute Convergence

If $U_1 + U_2 + U_3 + \ldots\ldots$ is Convergent with all terms positive, then a Series obtained by changing any of the signs to negative is also Convergent. The Series is called Absolutely Convergent

10 CALCULUS

Slope of a Curve

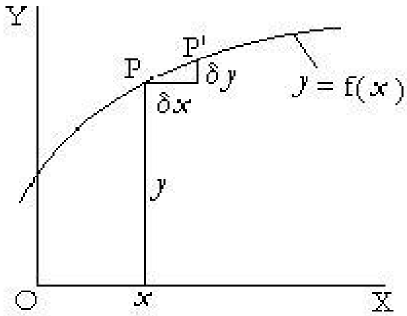


Figure 72: Slope of a curve

Consider the Curve $y = f(x)$ passing through two points P and P' which are close together.

The Slope of the Curve is $\delta y/\delta x$

At P $\quad y = f(x)$
At P' $\quad y + \delta y = f(x + \delta x)$

Therefore $\delta y = f(x + \delta x) - f(x)$

If P' is moved till almost at P, then $\delta y/\delta x$ is written as dy/dx

The Slope of the Curve at P = $\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Limit}} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$ (144)

Example

Let $y = ax^n$

Then $y + \delta y = a(x + \delta x)^n$

Expand by the Binominal theorem;

$y + \delta y = a\,[\,x^n + n\,x^{n-1}\,\delta x + \{n(n-1)/2!\}\,x^{n-2}\,\delta x^2 + \ldots\,]$

Therefore;

$\delta y = a\,[\,n\,x^{n-1}\,\delta x + \{n(n-1)/2!\}\,x^{n-2}\,\delta x^2 + \ldots\,]$

$\delta y/\delta x = a\,n\,x^{n-1} + a\,n(n-1)/2!\}\,x^{n-2}\,\delta x + \ldots$ higher powers of δx

dy/dx = Limit as $\delta x \to 0$ $[a\,n\,x^{n-1} + a\,n(n-1)/2!\}\,x^{n-2}\,\delta x + \ldots\,]$

$dy/dx = a\,n\,x^{n-1}$ (145)

dy/dx is known as the differential of y with respect to x

This is written as $d(a\,x^n)/dx = a\,n\,x^{n-1}$

Similarly, if $y = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \ldots$

Then the differential

$$dy/dx = 0 + a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + \ldots \quad (146)$$

Area under a Curve

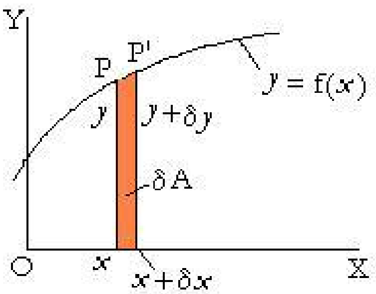


Figure 73: Area under a curve

Consider the small area δA between two points P and P'

Ignoring terms involving products of δx and δy,

$\delta A = y\,\delta x$

If P and P' are moved further apart, then Area A = Sum of all elemental Areas $y\,\delta x$
This is written as $A = \int y\,dx$
A is called the Integral of y with respect to x (147)

But $\delta A = y\,\delta x$
or $\delta A/\delta x = y$
This can be written as

$$\lim_{\delta x \to 0}\left(\frac{\delta A}{\delta x}\right) = y$$

Therefore $dA/dx = y$

Thus, if y is integrated with respect to x the result is A. If A is then differentiated with respect to x the result is again y. Thus integration is the inverse of differentiation.

Apply the rules of (145) to $y = a\,x^{n+1} + C$ where C is a constant
we can put the constant term as $C\,x^0$ since $x^0 = 1$

$$y = a\,x^{n+1} + C\,x^0$$
$$dy/dx = a\,(n+1)\,x^n + 0 \text{ times } C\,x^{-1}$$
$$= a\,(n+1)\,x^n$$

$$\int (n+1)\,a\,x^n\,dx = a\,x^{n+1} + C$$
$$\int a\,x^n\,dx = a\,x^{n+1}/(n+1) + C \qquad (148)$$

Integration introduces an unknown constant C. In the diagram, the integral is the area A up to point P. The value of A depends on the left hand edge which is not necessarily at the Origin.

Note also that if $n = -1$ the method fails as the differential of a constant is zero.

Integrating between Limits

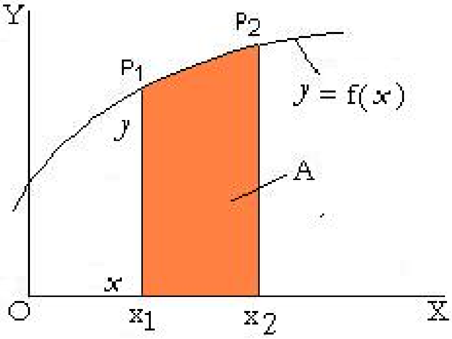


Figure 74: Integrating between Limits

The constant C is eliminated if both the left hand and right hand boundaries are defined. The shaded area in the diagram has boundaries at $x = x_1$ and $x = x_2$

Let $y = \mathrm{f}(x)$ and $\int y\,\mathrm{d}x = \mathrm{F}(x)$

Area A = value of $\mathrm{F}(x)$ between $x = x_1$ and $x = x_2$
$= [\ \mathrm{F}(x)$ with $x = x_2\ ] - [\ \mathrm{F}(x)$ with $x = x_1\ ]$
$=$ Integral $y\,\mathrm{d}x$ from $x = x_1$ to $x = x_2$

This is written as $\mathrm{A} = \int_{x1}^{x2} y\,\mathrm{d}x$

After Integration it is written as $\mathrm{A} = [\ \mathrm{F}(x)\ ]_{x1}^{x2}$ (149)

Example

Find the Area under the curve $y = 5x^2 + 3$ between $x = 2$ and $x = 3$

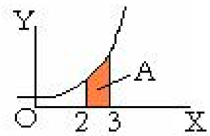


Figure 75: Area under $y = 5x^2 + 3$ between $x = 2$ and $x = 3$

$y = 5x^2 + 3$

$\int y \, dx = 5x^3/3 + 3x + C$

$A = \int_2^3 y \, dx = [5x^3/3 + 3x + C]_2^3$

$= [45 + 9 + C] – [13.33 + 6 + C] = 34.67$

Additional Rules for Differentiation

(i) Let $y = (u + v)$ then $y + \delta y = (u + \delta u + v + \delta v)$

Therefore $\delta y = (u + \delta u + v + \delta v) – (u + v) = \delta u + \delta v$

$\delta y/\delta x = \delta u/\delta x + \delta v/\delta x$

Therefore $dy/dx = du/dx + dv/dx$

$d(u + v)/dx = du/dx + dv/dx$ (150)

(ii) Let $y = u\,v$ then $y + \delta y = (u + \delta u)(v + \delta v) = uv + v\delta u + u\delta v + \delta u\delta v$

$\delta y = v\delta u + u\delta v + \delta u\delta v$

$\delta y/\delta x = v\delta u/\delta x + u\delta v/\delta x + \delta u\delta v/\delta x$

Taking the Limit as $\delta x \to 0$ $d(uv)/dx = v\,du/dx + u\,dv/dx$

$d(uv)/dx = v\,du/dx + u\,dv/dx$ (151)

(iii) Let $y = u / v$

$y = u / v = u\,v^{-1}$

$d(uv^{-1})/dx = v^{-1}du/dx + u\,d(v^{-1})/dx$

$= (1/v)\,du/dx + u\,[-1\,(v^{-2})]\,dv/dx$

$= [v\,du/dx – u\,dv/dx\,] / v^2$

$d(u/v)/dx = [v\,du/dx – u\,dv/dx\,] / v^2$ (152)

Change of variable

Let $y = F(u)$ and $u = f(x)$

$\delta y/\delta x = (\delta y/\delta u)\,(\delta u/\delta x)$

Taking the Limit as $\delta x \to 0$ $dy/dx = (dy/du)\,(du/dx)$

$dy/dx = (dy/du)\,(du/dx)$ (153)

Differentiation, summary

u and v are functions of x

$d/dx\,(u + v) = du/dx + dv/dx$ (differentiate a sum) (150)

$d/dx\,(u\,v) = v\,du/dx + u\,dv/dx$ (differentiate a product) (151)

$d/dx\,(u/v) = \{v\,du/dx - u\,dv/dx\}/v^2$ (differentiate a fraction) (152)

$dy/dx = (dy/du)(du dx)$ (called the chain rule) (153)

Any function can be differentiated using the rules (150) to (153)

Example (i)

$y = (a^2 + x^2)^3$ Find dy/dx

Put $u = (a^2 + x^2)$ therefore $y = u^3$

$dy/dx = dy/du\ du/dx$

$dy/du = 3u^2$ and $du/dx = 2x$ therefore $dy/dx = 6x(a^2 + x^2)^2$

Example (ii)

$y = (a^2 + x^2)^2\,(b - x)$ Find dy/dx

Put $u = (a^2 + x^2)^2$ and $v = (b - x)$ and $w = (a^2 + x^2)$

$u = w^2$ therefore $du/dx = du/dw\ dw/dx = 2w\,2x = 4x\,(a^2 + x^2)$

$dy/dx = u\,dv/dx + v\,du/dx$

$dv/dx = -1$ and $du/dx = 4x\,(a^2 + x^2)$

$dy/dx = (a^2 + x^2)^2\,(1) + (b - x)\,4x\,(a^2 + x^2) = (a^2 + x^2)\,(4bx - a^2 - 5x^2)$

Integration, summary

Integration may or may not be possible. Considerable ingenuity may be needed to put the expression in a form that can be integrated. Some functions cannot be integrated at all, although the value between limits can always be found, eg by plotting the curve and measuring the area under the curve.

Properties of e

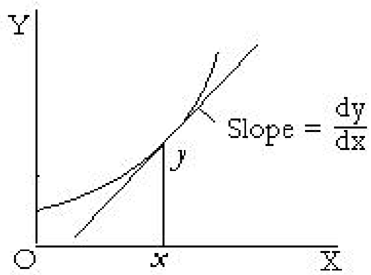


Figure 76: Slope proportional to y

There is a family of curves where the slope dy/dx is proportional to the value of y

$$dy/dx = a\,y \qquad (154)$$

The following method is one of the ways to evaluate this family of curves. The route seems circuitous to begin with but it finally arrives.

Consider the expression
$[\{1 + 1/n\}^n]^x$
$= [1 + 1/n]^{nx}$
$= 1 + nx(1/n) + nx(nx - 1)(1/n)^2/2! + nx(nx - 1)(nx - 2)(1/n)^3/3! + \dots$
$= 1 + x + x(x - 1/n)/2! + x(x - 1/n)(x - 2/n)/3! + \dots$

Thus the Limit as $n \to$ infinity of $[\{1 + 1/n\}^n]^x = 1 + x/1! + x^2/2! + x^3/3! + \dots..$

Put $x = 1$
The Limit as $n \to$ infinity of $\{1 + 1/n\}^n = 1 + 1/1! + 1/2! + 1/3! + \dots.$
This series is convergent and can be evaluated.
The series is called e and its value is approximately 2.718

Summarising;
e = Limit as $n \to$ infinity of $\{1 + 1/n\}^n$
$= 1 + 1/1! + 1/2! + 1/3! + \dots.$ (155)

Thus the Limit as $n \to$ infinity of $[\{1 + 1/n\}^n]^x = e^x$
Thus $e^x = 1 + x/1! + x^2/2! + x^3/3! + \dots..$ (156)

It can be seen that differentiating the series just brings each term one place to the left

Hence $\quad d/dx\,(e^x) = e^x$ (157)

Let $x = e^z$ then $\log_e x = z$

$dx/dz = d/dz\,(e^z) = e^z = x$

$\int (1/x)\,dx = \int dz = z + c = \log_e x + c$ where c is a constant

Log to base e (usually written ln) is called the Natural Logarithm

In these notes, ln means natural log. For other Logs, a base is given eg $\log_{10}$

$\int (1/x)\,dx = \ln x + c$ (158)

$d/dx\,[\ln x] = 1/x$ (159)

also $\quad d/dx\,[\ln (a\,x)] = d/dx\,[\ln x + \ln a] = 1/x$

Differentiation of e^{ax}

Put $a\,x = u$

$du/dx = a$

$d/dx\,(e^{ax}) = d/dx\,(e^u) = d/du(e^u)\,du/dx = e^u\,a$

Thus $d/dx\,(e^{ax}) = a\,e^{ax}$ (160)

Hence $\int (e^{ax})\,dx = (1/a)\,(e^{ax}) + c$ (161)

Differentiation of a^x

Let $a^x = z$ then $x = \log_a z$ From (131), $x = (\ln z) / (\ln a)$

Thus $\quad (\ln z) = x\,(\ln a)$ where $(\ln a)$ is a constant

Differentiate w.r.t x, $\quad (1/z)\,dz/d\,x = (\ln a)$

Therefore $dz/d\,x = z\,(\ln a) = a^x\,(\ln a)$

Thus $d/dx\,(a^x) = a^x(\ln a)$ (162)

Differentiation of Trigonometrical Functions

Let $y = \text{Sin } x$

$y + \delta y = \text{Sin } (x + \delta x) = \text{Sin } x \text{ Cos } \delta x + \text{Cos } x \text{ Sin } \delta x$ from (59)

$\approx \{1 - (\delta x)^2/2\} \text{ Sin } x + \delta x \text{ Cos } x$ from (58) and (56)

$\delta y = \delta x \text{ Cos } x - (\delta x)^2 / 2 \text{ Sin } x$

$\delta y/\delta x = \text{Cos } x - (\delta x / 2) \text{ Sin } x$

Therefore $dy/dx = \text{Cos } x$

Thus $d/dx (\text{Sin } x) = \text{Cos } x$ (163)

Similarly $d/dx (\text{Cos } x) = -\text{Sin } x$ (164)

$d/dx (\text{Tan } x) = d/dx (\text{Sin } x / \text{Cos } x)$

$= [\text{Cos } x \text{ Cos } x - \text{Sin } x (-\text{Sin } x)] / \text{Cos}^2 x$ from (152)

$= [\text{Cos}^2 x + \text{Sin}^2 x] / \text{Cos}^2 x$

$= 1 / \text{Cos}^2 x = \text{Sec}^2 x$

Thus $d/dx (\text{Tan } x) = \text{Sec}^2 x$ (165)

Let $y = \text{Arc Sin } (x/a)$ Therefore $\text{Sin } y = x/a$

Differentiate $(\text{Cos } y) \, dy/dx = 1/a$

Therefore;

$dy/dx = 1/(a \text{ Cos } y) = 1 / a\{\sqrt{(1 - \text{Sin}^2 y)}\} = 1/\sqrt{(a^2 - x^2)}$

ie $d/dx \{\text{Arc Sin } (x / a)\} = 1/\sqrt{(a^2 - x^2)}$ (166)

Similarly $d/dx \{\text{Arc Cos } (x/a)\} = -1/\sqrt{(a^2 - x^2)}$ (167)

And $d/dx \{\text{Arc Tan } (x/a)\} = a / (a^2 + x^2)$ (168)

11 NUMERICAL SOLUTION OF EQUATION

Solution by Computer

Equations of the form $f(x) = 0$ can be solved by a computer by trial and error. A value is assigned to x and and the value of $y = f(x)$ is calculated. A new value is assigned to x and the calculations repeated. The values of y are compared and if their signs are different, then there is a solution between them. The process is repeated with smaller steps between narrower limits. This is repeated again and again till y is close enough to zero for the required accuracy.

Simultaneous Equations

The values are assigned in steps to all variables except y in nested loops.
The value of $y = f(x_1, x_2, x_3, \text{etc})$ is evaluated for each equation. The process is repeated with smaller steps till values are found for all variables that satisfy the equations to the required accuracy.

Newton's Approximation

The solution to $f(x) = 0$ can be a slow process by trial and error.

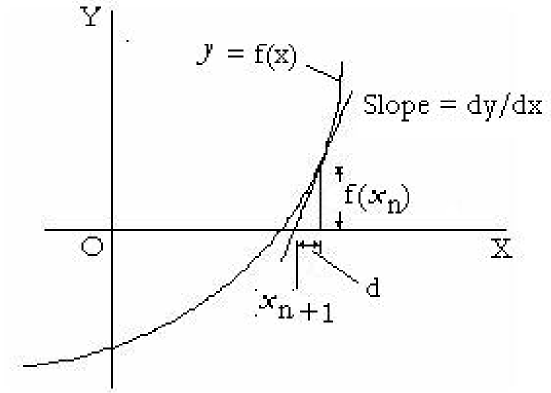


Figure 77: Newton's Apprpximation

However if the function $y = f(x)$ is differentiated, the solution can be obtained more quickly.

The slope of the curve $y = f(x)$
Let x_n be an approximate solution.
A closer approximation is $x_{n+1} = (x_n - d)$

Slope $dy/dx = f(x_n)/ d$

Therefore;
A closer approximation is $x_{n+1} = x_n - [f(x_n)/(dy/dx)_n]$

12 EXPANSION INTO A SERIES

MacLaurim's Theorem

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_r x^r + \ldots$

Write $f_r(x)$ to mean the r th differential of $f(x)$
$f_r(x) = f(x)$ differentiated r times

Write $f_r(0)$ for the value of the r th differential of $f(x)$ when $x = 0$
$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_r x^r + \ldots$
$f_1(x) = a_1 + 2 a_2 x + 3 a_3 x^2 + \ldots + r a_r x^{r-1} + \ldots.$
$f_2(x) = 2 a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \ldots + r(r-1) a_r x^{r-2} + \ldots$
$f_3(x) = 3.2.1 a_3 + 4.3.2 a_4 x + 5.4.3 a_5 x^2 + .+ r(r-1)(r-2) a_r x^{r-3} + \ldots$

$f_r(x) = r! a_r + (r+1)!/1! a_{r+1} x + (r+2)!/2! a_{r+2} x^2 + \ldots.$

Therefore $f(0) = a_0$ $f_1(0) = a_1$ $f_2(0) = 2! a_2$ $f_3(0) = 3! a_3$ $f_r(0) = r! a_r$ etc

MacLaurim's theorem,
$f(x) = f(0) + f_1(0) x/1! + f_2(0) x^2/2! + f_3(0) x^3/3! + \ldots. + f_r(0) x^r/r! +$ (169)

Expansion of Sin x and Cos x

Let $f(x) = \text{Sin } x$ Therefore $f(0) = 0$
$f_1(x) = \text{Cos } x$ $f_1(0) = 1$
$f_2(x) = -\text{Sin } x$ $f_2(0) = 0$
$f_3(x) = -\text{Cos } x$ $f_3(0) = -1$
$f_4(x) = \text{Sin } x$ $f_4(0) = 0$
etc

By Maclaurim's theorem;
$\text{Sin } x = 0 + x/1! + 0 - x^3/3! + 0 + x^5/5! + \ldots\ldots\ldots..$
$\text{Sin } x = x/1! - x^3/3! + x^5/5! - x^7/7!$ etc (170)

Similarly
$\text{Cos } x = 1 - x^2/2! + x^4/4! - x^6/6!$ etc (171)

Examples of Infinite Series

By Binominal Expansion;

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - ....$ provided $-1 < x < 1$ (172)

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$ provided $-1 < x < 1$ (173)

$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - ..$ .provided $-1 < x < 1$ (174)

$(1 + x)^{-1} (1 - x)^{-1} = (1 - x^2)^{-1} = 1 + x^2 + x^4 + x^6 + ...$ provided $-1 < x < 1$ (175)

Multiply (173) by x

$x/(1 - x) = x + x^2 + x^3 + x^4 + ...$ (176)

Integrating (172)

$\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 ....$ provided $-1 < x <= 1$ (177)

Integrating (173)

$\ln(1 - x) = - x - x^2/2 - x^3/3 - x^4/4 - ..$ provided $-1 < x < 1$ (178)

Integrating (174)

$\text{Arc Tan } x = x - x^3/3 + x^5/5 - x^7/7 + ...$ provided $-1 < x < 1$ (179)

Putting x = 0 shows that the constant of integration is zero in (177), (178) and (179).

Put $x = (1/3)$ in (172) $3/4 = 1 - 1/3 + 1/9 - 1/27 + ...$

Put $x = 1$ in (177) $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + ...$ (180)

Put $x = 1$ in (179) $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ (181)

Taylor's Theorem

If $f(x)$ and all its derivatives $f_1(x)$, $f_2(x)$ etc are all continuous in some range of x near $x = a$, then in this range;
$f(x) = f(a) + (x - a) f_1(a) + ... + \{(x - a)^r / r!\} f_r(a) + ...$

$$= \sum_{r=0}^{r=\infty} \frac{(x-a)^r}{r!} f_r(a) \qquad (182)$$

For proof, see Caunt page 465 or Lamb page 484

Alternatively as a Finite Series;
$f(x) = f(a) + (x - a) f(a) + ... + \{(x - a)^{n-1}/(n-1)!\} f_{n-1}(a) + R_n$
where $R_n = \{(x - a)^n/n!\} f_n(x_1)$ and x_1 is some value between x and a (183)

Write x instead of a and $x + h$ instead of x, Then $x_1 = x + \theta h$ where $0 < \theta < 1$

(183) becomes;

$$f(x + h) = \sum_{r=0}^{r=n-1} \left[\frac{h^r}{r!} f_r(x)\right] + \frac{h}{n!} f_n(x + \theta h) \qquad (184)$$

Putting $a = 0$ in (182) gives Maclaurim's Theorem.
Thus MacLaurim's Theorem is a particular case of Taylor's Theorem.

13 HYPERBOLIC FUNCTIONS

Properties of Cos θ + i Sin θ

Consider the Complex number Cos nθ + i Sin nθ
This expression is sometimes written as CiS nθ

Expand into a Series by (170) and (171)
$\text{Cos } n\theta + i \text{ Sin } n\theta = 1 - (n\theta)^2/2! + (n\theta)^4/4! - (n\theta)^6/6! +$
$+ i\,[(n\theta)/1! - (n\theta)^3/3! + (n\theta)^5/5! - \ldots$
But $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$ etc
Therefore
$\text{Cos } n\theta + i \text{ Sin } n\theta = 1 + (in\theta)/1! + (in\theta)^2/2! + (in\theta)^3/3! + (in\theta)^4/4! + (in\theta)^5/5! + .$
Therefore from (156)
$\text{Cos } n\theta + i \text{ Sin } n\theta = e^{in\theta}$

Therefore $\text{Cos } n\theta + i \text{ Sin } n\theta = e^{in\theta}$ (185)
Put n = 1 $\text{Cos } \theta + i \text{ Sin } \theta = e^{i\theta}$ (186)
Put n = –1 $\text{Cos } \theta - i \text{ Sin } \theta = e^{-i\theta} = 1/[\text{Cos } \theta + i \text{ Sin } \theta]$ (187)
Also $\text{Cos } n\theta + i \text{ Sin } n\theta = e^{in\theta} = [e^{i\theta}]^n = [\text{Cos } \theta + i \text{ Sin } \theta]^n$ (188)

Adding (186) and (187)
$\text{Cos } \theta = \{e^{i\theta} + e^{-i\theta}\}/2$ (189)
Subtracting (187) from (186)
$\text{Sin } \theta = \{e^{i\theta} - e^{-i\theta}\}/2i$ (190)

Hyperbolic Functions

By definition Sinh and Cosh are the same as Sin and Cos but without the complex number i
$\text{Sinh } \theta = \{e^{\theta} - e^{-\theta}\}/2 = \theta/1! + \theta^3/3! + \theta^5/5! + \ldots.$ (191)
(usually pronounced Shine)
$\text{Cosh } \theta = \{e^{\theta} + e^{-\theta}\}/2 = 1 + \theta^2/2! + \theta^4/4! + \ldots.$ (192)
$\text{Tanh } \theta = (\text{Sinh } \theta)/(\text{Cosh } \theta) = \{e^{\theta} - e^{-\theta}\}/\{e^{\theta} + e^{-\theta}\}$ (193)
(usually pronounced Than)

Sech θ = 1/Cosh θ (usually pronounced Sheck) (194)
Cosech θ = 1/Sinh θ (usually pronounced Cosheck) (195)
Coth θ = 1/Tanh θ (196)

Properties of Hyperbolic Functions

Adding (191) and (192);
$\text{Sinh } \theta + \text{Cosh } \theta = e^{\theta}$ (197)

Similarly

$\text{Cosh } \theta - \text{Sinh } \theta = e^{-\theta}$ (198)

$\text{Sinh}^2 \theta = (1/4)(e^{2\theta} - 2e^{\theta}e^{-\theta} + e^{-2\theta})$

$= (1/4)(e^{2\theta} - 2 + e^{-2\theta})$

Similarly

$\text{Cosh}^2 \theta = (1/4)(e^{2\theta} + 2 + e^{-2\theta})$

Therefore

$\text{Cosh}^2 \theta - \text{Sinh}^2 \theta = 1$` (199)

Divide by $\text{Cosh}^2 \theta$

$1 - \text{Tanh}^2 \theta = \text{Sech}^2 \theta$ (200)

Also

$\text{Sinh } i\theta = \{e^{i\theta} - e^{-i\theta}\} / 2 = i \text{ Sin } \theta$ (201)

$\text{Cosh } i\theta = \{e^{i\theta} + e^{-i\theta}\} / 2 = \text{Cos } \theta$ (202)

Put $i\theta = x$

$\text{Sinh } x = i \text{ Sin } (x/i) = -i \text{ Sin } (i x)$ (203)

$\text{Cosh } x = \text{Cos } (x/i) = \text{Cos } (i x)$ (204)

$\text{Tanh } x = -i \text{ Tan } (i x)$ (205)

$\text{Sinh } 2x = -i \text{ Sin } (2ix) = -i\, 2 \text{ Sin } (ix) \text{ Cos } (ix)$

$= 2 \text{ Sinh } x \text{ Cosh } x$ (206)

$\text{Cosh } 2x = \text{Cos } (2ix) = \text{Cos}^2 (ix) - \text{Sin}^2 (ix)$

$= \{\text{Cos } (ix)\}^2 + \{-i \text{ Sin } (ix)\}^2$

$= \text{Cosh}^2 x + \text{Sinh}^2 x$ (207)

Therefore $\text{Cosh}^2 x = ½ (\text{Cosh } 2x + 1)$ (208)

And $\text{Sinh}^2 x = ½ (\text{Cosh } 2x - 1)$ (209)

Similarly

$\text{Sinh } (A + B) = \text{Sinh } A \text{ Cosh } B + \text{Cosh } A \text{ Sinh } B$ (210)

$\text{Cosh } (A + B) = \text{Cosh } A \text{ Cosh } B + \text{Sinh } A \text{ Sinh } B$ (211)

Differentiation of Hyperbolic Functions

Let $y = \text{Sinh } x = -i \text{ Sin } i x$

Therefore $dy/dx = -i(\text{Cos } i x) . i = \text{Cos } i x = \text{Cosh } x$

$d/dx (\text{Sinh } x) = \text{Cosh } x$ (212)

Let $y = \text{Cosh } x = \text{Cos } i x$

Therefore $dy/dx = -(\text{Sin } i x) .i = \text{Sinh } i x$

$d/dx (\text{Cosh } x) = \text{Sinh } x$ (213)

Let $y = \text{Tanh } x = -i \text{ Tan } i x$

Therefore $dy/dx = -i (\text{Sec}^2 i x)\, i = \text{Sec}^2 i x = \text{Sech}^2 x$

$d/dx (\text{Tanh } x) = \text{Sech}^2 x$ (214)

Let $y = \text{Arc Sinh } (x/a)$ Therefore $\text{Sinh } y = x/a$

Differentiate

$(\text{Cosh } y)\, dy/dx = 1/a$

$$\frac{d}{dx}[\text{Arc Sinh}\,(x/a)] = dy/dx = 1/(a \cosh y) = 1/[a\sqrt{(1 + \sinh^2 y)}]$$
$$= 1/\sqrt{(a^2 + x^2)} \qquad (215)$$

Let $y = \text{Arc Cosh}\,(x/a)$ Therefore $\cosh y = x / a$

Differentiate

$(\sinh y)\, dy/dx = 1/a$

$$d/d x\,[\text{Arc Cosh}\,(x/a)] = dy/d x = 1/(a \sinh y) = 1/[a /(\cosh^2 y - 1)]$$
$$= 1/\sqrt{(x^2 - a^2)} \qquad (216)$$

Let $y = \text{Arc Tanh}\,(x/a)$ Therefore $\tanh y = x/a$

Differentiate

$[\text{Sech}^2 y]\, dy/dx = 1/a$

From (200)

$(1 - \tanh^2 y)\, dy/dx = 1/a$

$(1 - x^2/a^2)\, dy/d x = 1/a$

$$d/dx\,[\text{Arc Tanh}\,(x/a)] = a/(a^2 - x^2) \qquad (217)$$

14 METHODS FOR INTEGRATION

Integration by Standard Form

If the Integral can be written in the form of any of the expressions in the first or second column, it can be integrated at once.

y	dy / dx	$\int y\, dx$
$a\, x^n$	$n\, a\, x^{n-1}$	$a\, x^{n+1} /(n + 1)$
a / x	$-a / x^2$	$a \ln x$
$\text{Sin}\, (\omega x)$	$\omega \text{Cos}\, (\omega x)$	$(-1/\omega)\, \text{Cos}\, (\omega\, x)$
$\text{Cos}\, (\omega\, x)$	$-\omega \text{Sin}\, (\omega\, x)$	$(1/ \omega)\, \text{Sin}\, (\omega\, x)$
$\text{Tan}\, (\omega\, x)$	$\omega \text{Sec}^2\, (\omega\, x)$	$-(1/ \omega) \ln [\text{Cos}\, (\omega\, x)]$
$\text{Sec}\, x$	$\tan x\;\; \text{Sec}\, x$	$\ln (\text{Sec}\, x + \text{Tan}\, x)$
Cosec x	– Cot x Cosec x	ln (Cosec x – Cot x)
$\text{Cot}\, x$	$-\text{Cosec}^2\, x$	$\ln (\text{Sin}\, x)$
$\text{Arc Sin}\, (x /a)$	$1 /\sqrt{}\, (a^2 - x^2)$	$x\, \text{Arc Sin}\, (x /a) + \sqrt{}\, (a^2 - x^2)$
$\text{Arc Cos}\, (x /a)$	$-1 /\sqrt{}\, (a^2 - x^2)$	$x\, \text{Arc Cos}\, (x /a) - \sqrt{}\, (a^2 - x^2)$
$\text{Arc Tan}\, (x /a)$	$a / (a^2 + x^2)$	$x\;\; \text{Arc Tan}\, (x/a) - a \ln\sqrt{}(a^2 + x^2)$
e^{ax}	$a\, e^{ax}$	$(1/a)\, e^{ax}$
a^x	$a^x \ln(a)$	$a^x/[\ln (a)]$
$\ln (a\, x)$	$1/x$	$x\;\; \ln (a\, x - 1)$
$\text{Log}_a\, x$	$(1/x)\, \text{Log}_a\, e$	$x\, \text{Log}_a\, (x/e)$
$\text{Sinh}\;\; x$	$\text{Cosh}\, x$	$\text{Cosh}\, x$
$\text{Cosh}\, x$	$\text{Sinh}\, x$	$\text{Sinh}\, x$
$\text{Tanh}\, x$	$\text{Sech}^2\, x$	$\ln (\text{Cosh}\, x)$
$\text{Arc Sinh}\, (x /a)$	$1/\sqrt{}\, (a^2 + x^2)$	$x\, \text{Arc Sinh}\, (x/a) - \sqrt{}(a^2 + x^2)$
$\text{Arc Cosh}\, (x/a)$	$1/\sqrt{}\, (x^2 - a^2)$	$x\, \text{Arc Cosh}\, (x/a) - \sqrt{}(x^2 - a^2)$
$\text{Arc Tanh}\, (x/a)$	$a/(a^2 - x^2)$	$x\, \text{Arc Tanh}\, (x/a) + a \ln\sqrt{}(a^2 - x^2)$

(218)

Change of variable

Look for a Substitution that will simplify the Integral
Substitute $u = f(x)$ to convert the integral to a standard form. (219)

Examples
$I = \int F(a\,x + b)\,dx$ Put $u = a\,x + b$ Therefore $du = a\,dx$ and $I = (1/a)\int F(u)\,du$
$I = \int F(a\,x^2 + b)\,x\,dx$ Put $u = a\,x^2 + b$ Therefore $du = 2ax\,dx$ and $I = (1/2a)\int F(u)\,du$
$I = \int [F(x^2)]/\,x\,dx = \int [F(x^2)\,/\,x^2]\,x\,dx$ Put $u = x^2$ and $I = \int [F(u)]\,/u\,du$

Partial Fractions

Integrals of Fractions, eg $I = \int [\,f(x)\,/\,F(x)\,]\,dx$
Divide out and put the Remainder into Partial Fractions

$f(x)/F(x) = a_0 + a_1x + a_2x^2 + \ldots + A/(x + \alpha) + B/(x + \beta) + C/(x + \gamma) + ..$ etc
Integrate to;
$a_0\,x + (1/2)\,a_1x^2 + (1/3)a_2\,x^3 + \ldots + A\ln(x + \alpha) + B\ln(x + \beta) + C\ln(x + \gamma)$ etc (220)

Example (i)
$I = \int[1/(x^2 - a^2)]\,dx$
Put into Partial Fractions $1/(x^2 - a^2) = A/(x + a) + B/(x - a)$
Multiply by $(x - a)$
$1/(x + a) = A(x - a)/(x + a) + B$
This is true for all values of x, put $x = a$ and $B = 1/2a$. Similarly $A = 1/2a$
$I = \int[1/(x^2 - a^2)]\,dx$
$= \int[(1/2a)/(x - a)]dx - \int[(1/2a)/(x + a)]dx$
$= (1/2a)[\ln(x - a) - \ln(x + a)] + \text{constant}$ (221)

Example (ii)
$I = \int[1/(a^2 - x^2)]\,dx$
Put into Partial Fractions $A/(a + x)$ and $B/(a - x)$ (222)

Trigonometry Substitutions

$\int[1/(a^2 + x^2)]\,dx$ does not factorize
It cannot therefore be put into Partial Fractions. Look for a substitution that simplifies the Integral
This suggests $x = a \text{ Tan } u$ (or $x = \text{Sinh } u$)
Try $x = a \text{ Tan } u$ therefore $dx = a \text{ Sec}^2 u\, du$
$\int[1/(a^2 + x^2)]\,dx = (1/a^2)\int(a \text{ Sec}^2 u\, du) / (1 + \tan^2 u)$
$= (1/a)\int(\sec^2 u\, du) / (\sec^2 u) = (1/a)\int du = u/a = (1/a) \text{ Arc tan } (x/a)$ (223)

Integrals with $ax^2 + bx + c$ as the denominator

If $(ax^2 + bx + c)$ factorizes, then split into Partial Fractions as above

If it does not factorize, then remove the x term
Put $u = x + b/2a$ and $A^2 =$ positive value of $\pm [c/a - (b^2/4a^2)]$
Therefore $1/[a x^2 + b x + c] = 1/[a (u^2 \pm A^2)]$

If the numerator is $u\,du$, Integrate at once leading to $\ln(u^2 \pm A^2)$ etc

If the numerator is du and $u^2 \pm A^2$ has the positive sign, put $v = A \tan u$
If $u^2 \pm A^2$ has the negative sign, split into Partial Fractions with denominators $(u + A)$ and $(u - A)$
If $A = 0$, Integrate at once to u^{-1}

Functions of Square Roots can often be Integrated after a Trigonometrical Substitution.
Look for a substitution that will remove the square root.

$\int F[\sqrt{(a^2 - x^2)}]\,dx$ suggests a substitution $x = a \text{ Sin } u$ (224)
$\int F[\sqrt{(a^2 + x^2)}]\,dx$ suggests a substitution $x = a \text{ Sinh } u$ (225)
$\int F[\sqrt{(x^2 - a^2)}]\,dx$ suggests a substitution $x = a \text{ Cosh } u$ (226)

Integrals with $\sqrt{(ax^2 + bx + c)}$ as the denominator

$\int[1/\sqrt{(ax^2 + bx + c)}]dx$
Remove the x term, Put $a[(x + p)^2 + q] = ax^2 + bx + c$
$a[x^2 + 2px + p^2 + q] = ax^2 + bx + c$
Equate coefficients to solve for p and q
$p = b/2a$ and $q = c/a - p^2$
Put $u = x + p$ and $r^2 = q$
This leads to $I = (1/\sqrt{a}) \int[1/\sqrt{(u^2 \pm r^2)}]dx$
As above, if denominator is $\sqrt{(u^2 + r^2)}]$ then put $u = r \text{ Sinh } v$ (227)
If denominator is $\sqrt{(u^2 - r^2)}]$ then put $u = r \text{ Cosh } v$ (228)
If $r = 0$, then the integral $= (1/\sqrt{a})\log u +$ constant (229)

Integrals of Trigonomety Functions

(i) If possible, put in the form $I = \int(F(u)\, du$
such as $\int(F(\text{Cos } x) \text{ Sin } x\, dx$
or $\int(F(\text{Sin } x) \text{ Cos } x\, dx$
or $\int(F(\text{Tan } x) \text{ Sec}^2 x\, dx$
or $\int(F(\text{Sinh } x) \text{ Cosh } x\, dx$
or $\int(F(\text{Cosh } x) \text{ Shinh } x\, dx$

For example
$I = \int(\text{Sinh}^3 x)\, dx = \int(\text{Cosh}^2 x - 1) \text{ Sinh } x)\, dx = 1/3 \text{ Cosh}^3 x - \text{Cosh } x + c$

(ii) Try the Substitution $u = \text{Tan } (x)$ since $dx = du/(1 + u^2)$

For example
$I = \int [1 / (a^2 \text{ Cos}^2 x + b^2 \text{ Sin}^2 x)]\, dx$
Put $u = \text{Tan } x$ therefore $du = \text{Sec}^2 x\, dx$ and $dx = \text{Cos}^2 x\, du$
$I = \int [1 / (a^2 \text{ Cos}^2 x + b^2 \text{ Sin}^2 x)] \text{ Cos}^2 x\, du$
$= \int [1 / (a^2 + b^2 \text{ Tan}^2 x)]\, du$
$= \int [1 / (a^2 + b^2 u^2)]\, du$
Put $u = (a/b) \text{ Tan } v$ therefore $du = (a/b) \text{ Sec}^2 v\, dv$
$I = \int [1/\{a^2 (1 + \text{Tan}^2 v)\}] (a/b) \text{ Sec}^2 v\, dv = (1/ab) \int dv = (1/ab)\, v +$ const
$I = (1/ab) \text{ Tan}^{-1} [(b/a) \text{ Tan } x] +$ const

(iii) Try the Substitution $t = \mathrm{Tan}(x/2)$
$dx = 2\,dt/(1+ t^2)$, $\sin x = 2t/(1+ t^2)$ and $\mathrm{Cos}\, x = (1 - t^2)/(1+ t^2)$
All have the same denominator which may cancel out.
For example $\int[1/(a \,\mathrm{Sin}\, x + b \,\mathrm{Cos}\, x + c)]dx$ indicates the substitution $t = \mathrm{Tan}(x/2)$
$I = \int[1/\{a2t + b(1 - t^2)+ c(1+ t^2)\}]\ 2\,dt$
$I = \int[1/\{(c - b)\, t^2 + 2a\, t + (b + c)\}]\ 2\,dt$
Split into Partial Fractions or remove the t term as above

$\int F(\mathrm{Sin}^2 x)\, dx$ and $\int F(\mathrm{Cos}^2 x)\, dx$

$\int(\mathrm{Sin}^2 x)\, dx$ and $\int(\mathrm{Cos}^2 x)\, dx$ indicate the substitution $u = 2x$ since from equation (68)
$\mathrm{Cos}^2(x) = ½\,[\mathrm{Cos}\,(u) + 1]$ and $dx = ½\, du$
For example $\int a^2 \,\mathrm{Sin}^2 (x)\ dx = a^2 \int[1 - \mathrm{Cos}^2 (x)]\, dx = a^2 \int[1 - ½\,\{\mathrm{Cos}\,(u) + 1\}]\ ½\, du$
$= ¼\,(a^2 u) - ¼[a^2 \,\mathrm{Sin}\,(u)] + \text{constant}$

$\int a^2 \,\mathrm{Sin}^2 (x)\ dx$ from 0 to 2π (ie $u = 0$ to 4π) is $a^2 \pi$
Hence Average value of $a^2 \,\mathrm{Sin}^2 (x)$ is $a^2 /2$

Integration by Parts

From (151) $d/dx\,(u\,v) = u\,dv/dx + v\,du/dx$
Integrate with respect to x; $u\,v = \int udv + \int vdu$
Rearrange $\int udv = uv - \int vdu$

Use this formula to transform the integration of a product. (230)

Example (i) $I = \int x \,\mathrm{Sin}\, x\, dx$
Put $u = x$, $dv = \mathrm{Sin}\, x\, dx$ therefore $v = -\mathrm{Cos}\, x$
$I = -x\,\mathrm{Cos}\, x + \int \mathrm{Cos}\, x\, dx = -x\,\mathrm{Cos}\, x + \mathrm{Sin}\, x + \text{constant}$
Example (ii) $I = \int x \ln(x)\, dx$
Put $u = \ln x$, $dv = x\,dx$ therefore $v = (1/2)\,x^2$
$I = (1/2)\,x^2 \ln(x) - \int (1/2)\,x^2\,(1/x)\,dx = (1/2)\,x^2 \ln(x) - \int (1/2)\,x\,dx$
$= (1/2)\,x^2 \ln(x) - (1/4)\,x^2 + \text{constant}$

1/D method

This gives a simpler solution than Integration by Parts for some expressions, such as $e^{ax} f(x)$

By definition, D is an operator that differentiates the expression after it.

Therefore $D = d/dx$ and $D(y) = dy/dx$ (231)
$D\,(D(y)) = D^2(y) = d^2 y/dx^2$ (232)
$D(y_1 + y_2) = d/dx\,(y_1 + y_2) = dy_1/dx + dy_2/dx = D(y_1) + D(y_2)$ (233)
$D^m D^n (y) = d^m/dx^m\,(d^n y/dx^n) = d^{m+n} y/dx^{m+n} = D^{m+n}(y)$ (234)
If c is a constant, $D(c\,y) = d/dx\,(c\,y) = c dy / dx = c\,D(y)$ (235)

If u and v are variables,
$D(uv) = d/dx(uv) = udv/dx + vdu/dx = u\,D(v) + v\,D(u)$ (236)

Thus D satisfies most or the rules of algebra except that the order of D and a variable cannot be changed.
Write $(D + 2)y$ to mean $D(y) + 2\,y = dy/dx + 2\,y$
$(D - 3)y$ $D(y) - 3\,y = dy/dx - 3\,y$
$(D^2 - D - 6)\,y = D^2(y) - D(y) - 6\,y = d^2y/dx^2 - dy/dx - 6\,y$

Consider two successive operations;
First perform $(D + 2)$ on y to get z, therefore $z = dy/dx + 2\,y$
Second perform $(D - 3)$ on z to get $dz/dx - 3z = d^2y/dx^2 + 2\,dy/dx - 3\,d\,y\,/dx - 6\,y$
Therefore $(D + 2)\,(D - 3)\,y = d^2y/dx^2 - dy/dx - 6\,y = (D^2 - D - 6)\,y$
Thus the operator D can be multiplied or factorised in the usual way.

It has been shown that $D^m D^n (y) = D^{m+n} (y)$
But $D(\int y\, dx) = y$ ie $y = D(\int y\, dx)$

Operate D^{-1} on both sides
$D^{-1}(y) = D^{-1}(D(\int y\, dx)) = D^0(\int y\, dx) = \int y\, dx$
Therefore $D^{-1}(y) = \int y\, dx$ (237)
Let V be any function of x
Consider the differentiation of the product $e^{ax} V$;

$$D(e^{ax} V) = d/dx(e^{ax} V) = a e^{ax} V + e^{ax} dV/dx$$
$$= e^{ax} (dV/dx + a V)$$
$$= e^{ax} (D + a) V$$
$$= e^{ax} V_1 \quad \text{where } V_1 = (D + a) V$$

Differentiate with respect to x

$$D^2 (e^{ax} V) = D(e^{ax} V_1) = e^{ax} (D + a) V_1 = e^{ax} (D + a)^2 V$$

In general $D^n (e^{ax} V) = e^{ax} (D + a)^n V$ (238)

Consider $I = \int e^{ax} V\, dx$
$I = D^{-1} (e^{ax} V)$

Assume (238) is still true for negative values of n

$$I = e^{ax} (D + a)^{-1} V = (1/a) e^{ax} (1 + D/a)^{-1} V$$
$$= (1/a) e^{ax} \{1 - (D/a) + (D/a)^2 - (D/a)^3 + \ldots\} V$$
$$\int e^{ax} V\, dx = (1/a) e^{ax} \{1 - (D/a) + (D/a)^2 - (D/a)^3 + \ldots\} V \quad (239)$$

Example

$$I = \int e^{ax} (ax^2 + bx + c)\, dx$$
$$= (1/a) e^{ax} (1 - (D/a) + (D/a)^2 - (D/a)^3 + \ldots) (a x^2 + b x + c)$$
$$= (1/a) e^{ax} [(a x^2 + b x + c) - (1/a)(2 a x + b) + (1/a)^2 (2a)]$$
$$= e^{ax} (a^2 x^2 + a b x + a c - 2 a x - b + 2) / a^2$$

Check the result;

$$dI/dx = a e^{ax} (a^2x^2 + abx + ac - 2ax - b + 2)/a^2 + e^{ax}(2a^2x + ab - 2a)/a^2$$
$$= e^{ax} (ax^2 + bx + c - 2x - b/a + 2/a + 2x + b/a - 2/a)$$
$$= e^{ax} (ax^2 + bx + c)$$

It follows from (238) that;

$$F(D) \{e^{ax} V\} = e^{ax} F(D + a) V \quad (240)$$

Putting $V = 1$;

$$F(D) e^{ax} = e^{ax} F(a) \quad (241)$$

If the theorems still hold with D in the denominator, then from (240)

$$[1 / F(D)]\ [e^{ax} V] = e^{ax}\ [1/ F(D + a)]\ V \qquad (242)$$

Putting V = 1;

$$[1 / F(D)]\ e^{ax} = [1 / F(a)]\ e^{ax} \qquad (243)$$

Consider the operator D acting on Sin (mx)

$$\begin{aligned} D(\text{Sin } mx) &= m \text{ Cos } mx \\ D^2(\text{Sin } mx) &= - m^2 \text{ Sin } mx \\ D^3(\text{Sin } mx) &= - m^3 \text{ Cos } mx \\ D^4(\text{Sin } mx) &= m^4 \text{ Sin } mx \end{aligned}$$

Similarly for the operator D acting on Cos mx

Thus D^2 can be replaced by $-m^2$

$$F(D^2)\ (a \text{ Sin } mx + b \text{ Cos } mx) = F(-m^2)\ (a \text{ Sin } mx + b \text{ Cos } mx) \qquad (244)$$

Example

$$\begin{aligned} I &= \int [e^{ax} \{A \text{ Sin}(mx) + B \text{ Cos}(mx)\}]\, dx \\ &= e^{ax} (D + a)^{-1} (A \text{ Sin } mx + B \text{ Cos } mx) \\ &= e^{ax} (D - a)\ (D - a)^{-1}\ (D + a)^{-1} (A \text{ Sin } mx + B \text{ Cos } mx) \\ &= e^{ax} (D - a)\ (D^2 - a^2)^{-1}\ (A \text{ Sin } mx + B \text{ Cos } mx) \\ &= e^{ax} (D - a)\ (-m^2 - a^2)^{-1}\ (A \text{ Sin } mx + B \text{ Cos } mx) \\ &= -(e^{ax}) / (m^2 + a^2)\ (D - a)\ (A \text{ Sin } mx + B \text{ Cos } mx) \\ &= -(e^{ax})/(m^2 + a^2)\ (Am \text{ Cos } mx - Bm \text{ Sin } mx - aA \text{ Sin } mx - aB \text{ Cos } mx) \end{aligned}$$

CiS(x) method

The above example demonstrates the use of the D method.
However there is another method by considering $\text{Cos } mx + i \text{ Sin } mx$
$I = \int [e^{ax} \{A \text{ Sin } (mx) + B \text{ Cos}(mx)\}] \, dx$
The Integral is the Real part of $\int [e^{ax} B\{\text{Cos}(mx) + i \text{ Sin } (mx)\}] \, dx$
plus the Complex part of $\int [e^{ax} A\{\text{Cos}(mx) + i \text{ Sin } (mx)\}] \, dx$
From (185) $\text{Cos } mx + i \text{ Sin } mx = e^{imx}$
Consider the Real part of $\int [e^{ax} B\{\text{Cos}(mx) + i \text{ Sin } (mx)\}] \, dx$
= Real part of the Complex Integral $\int [e^{ax} B \, e^{imx}] \, dx$
= Real part of Integral $B \int [e^{(a+im)x}] \, dx$
$= B \, [e^{(a+im)x}] \, / \, (a + im)$
$= B \, e^{ax} (\text{Cos } mx + i \text{ Sin } mx) \, /(a + im)$
$= B \, e^{ax} (\text{Cos } mx + i \text{ Sin } mx) \, (a - im)/(a^2 + m^2)$
Real part $= B \, e^{ax} \, a/ \, (a^2 + m^2) \text{ Cos } mx + B \, e^{ax} \, m/ \, (a^2 + m^2) \text{ Sin } mx$
$= B \, e^{ax}[\, a \text{ Cos } mx + m \text{ Sin } mx \,] \, /(a^2 + m^2)$ (245)

Integral $I = \int [e^{ax} A \text{ Sin } (mx) \, dx$
= Complex part of $\int [e^{ax} A \, \{\text{Cos } (mx) + i \text{ Sin}(mx)\}] \, dx$
= Complex part of Integral $\int [e^{ax} A \, e^{imx}] \, dx$
= Complex part of Integral $A \int [e^{(a+im)x}] \, dx$
= Complex part of $A \, [e^{(a+im)x}] \, / \, (a + im)$
= Complex part of $A \, e^{ax} (\text{Cos } mx + i \text{ Sin } mx) \, (a - im)/ \, (a^2 + m^2)$
= Complex part of$A \, e^{ax} [- i \, m \text{ Cos } mx + i \, a \text{ Sin } mx \,] \, /(a^2 + m^2)$ (246)

Therefore $I = \int [e^{ax} \{A \text{ Sin } (mx) + B \text{ Cos}(mx)\}] \, dx$
$= e^{ax} [(Aa + Bm) \text{ Sin } mx + (Ba - Am)\text{Cos}(mx)] \, /(a^2 + m^2)$
This is the same as the result by the D method.

Irrational Functions

An Irrational function is a function involving square roots, cube roots etc.
Substitutions as above may work, otherwise try;

$I = \int F(x, Y)$ where $Y = \sqrt[m]{(a + b \, x)}$
Put $(a + bx) = t^m$
Therefore $dx = m \, t^{m-1} \, /b \, dt$
$I = \int F[(t^m - a)/b, t] \, m \, t^{m-1} \, /b \, dt$ (247)

Integrals of trigonometry functions between 0 and π/2

$$I = \int_0^{\frac{\pi}{2}} \text{Sin}^m x \text{ Cos}^n x \, dx$$

write $S = \text{Sin } x$ and $C = \text{Cos } x$
Consider $d/dx \, (S^{m+1} \, C^{n-1}) = (m + 1) \, S^m \, C^n - (n - 1) \, S^{m+2} \, C^{n-2}$
$= (m + 1) \, S^m \, C^n - (n - 1) \, S^m \, C^{n-2} \, (1 - C^2)$
$= (m + n) \, S^m \, C^n - (n - 1) \, S^m \, C^{n-2}$

Integrate with respect to x from 0 to $\pi/2$

$$[s^{m+1}\ c^{n-1}]_0^{\frac{\pi}{2}} = (m+n)\int_0^{\frac{\pi}{2}} s^m c^n dx - (n-1)\int_0^{\frac{\pi}{2}} s^m\ c^{n-2}\, dx$$

$S = 0$ if $x = 0$, and $C = 0$ if $x = \pi/2$

Therefore $[s^{m+1}\ c^{n-1}]^{\frac{\pi}{2}} = 0$

And

$$\int_0^{\frac{\pi}{2}} s^m c^n dx = \frac{(n-1)}{(m+n)}\int_0^{\frac{\pi}{2}} s^m\ c^{n-2}\, dx \qquad (248)$$

Similarly

$$\int_0^{\frac{\pi}{2}} s^m c^n dx = \frac{(m-1)}{(m+n)}\int_0^{\frac{\pi}{2}} s^{m-2}\ c^n\, dx \qquad (249)$$

Example

$$\int_0^{\frac{\pi}{2}} \mathrm{Sin}^3 x\ \mathrm{Cos}^5 x\ dx = \frac{3-1}{3+5}\int_0^{\frac{\pi}{2}} \mathrm{Sin}\ \mathrm{Cos}^5 x\ dx$$

$$= \frac{2}{8}\left[-\frac{1}{6}\mathrm{Cos}^6 x\right]_0^{\frac{\pi}{2}} = \frac{2}{8}\ \frac{1}{6}(-1)^6 = \frac{1}{24}$$

General Reduction Formula

$I = \int[\sin^m (x) \cos^n (x)]\, dx$ between any limits

As above $d/dx\,(S^{m+1}\ C^{n-1}) = (m + n)\, S^m C^n - (n - 1)\, S^m C^{n-2}$

$S^m C^n = [1/(m + n)]\, d/dx\,(S^{m+1}\ C^{n-1}) + [(n - 1)/(m + n)]\, S^m C^{n-2}$

Integrate

$\int[\sin^m (x) \cos^n (x)]\, dx = [1/(m + n)]\,(\sin^{m+1} \cos^{n-1}) + [(n - 1)/(m + n)] \int[\sin^m (x) \cos^{n-2} (x)]\, dx$ (250)

Also $d/dx\,(S^{m-1}\ C^{n+1}) = (m - 1)\, S^{m-2} C^{n+2} - (n +1)\, S^m\ C^n$

$= (m - 1)\, S^{m-2} C^n(1 - S^2) - (n +1)\, S^m\ C^n$

$= (m - 1)\, S^{m-2} C^n - (m - 1)\, S^m C^n - (n +1)\, S^m\ C^n$

$= (m - 1)\, S^{m-2} C^n - (m + n)\, S^m C^n$

$(m + n)\, S^m C^n = (m - 1)\, S^{m-2} C^n - d/dx\,(S^{m-1}\ C^{n+1})$

Integrate

$\int[\sin^m (x) \cos^n (x)]\, dx = [(m - 1)/(m + n)] \int[\sin^{m-2} (x) \cos^n (x)]\, dx - [1/(m + n)](\sin^{m-1} \cos^{n+1})$ (251)

15 FUNCTIONS OF TIME AND OTHER VARIABLES

Functions of time

Let $x = F(t)$ be the distance of an object from a fixed point at time t

Then $dx / dt = v$ the velocity away from the fixed point (252)

And $d^2x / dt^2 = a$ the acceleration away from the fixed point (253)

dx/dt, the velocity, is sometimes written as x dot $\dot{X}$

d^2x/dt^2 , the acceleration, is sometimes written as x double dot $\ddot{X}$

Note that $a = d/dt\,(dx/dt) = dv/dt = (dv/dx)\,(dx/dt)$

Thus $a = v\,dv / dx$ (254)

Use this result for problems where the velocity is related to distance rather than time.

$x = \int v \, dt + c$ (255)

$v = \int a \, dt + c$ (256)

Let θ be the angle in radians of rotation from a fixed point (257)

the angular speed of rotation $d\theta / dt = \omega$ which is sometimes written as theta dot $\dot{\theta}$ (258)

the angular acceleration $d^2\theta / dt^2 = d\omega / dt$ = theta double dot $\ddot{\theta}$ (259)

$\theta = \int \omega \, dt + c$ (260)

Example
Simple Harmonic Motion $x = A \sin(\omega t)$
Find the velocity when $x = 0$ and the acceleration when $x = A$

Velocity $v = dx / dt = A \omega \cos(\omega t)$
$x = 0$ when $(\omega t) = 0$. Therefore when $x = 0$, the velocity $= A \omega$

Acceleration $a = dv / dt = - A \omega^2 \sin(\omega t)$
$x = A$ when $(\omega t) = \pi / 2$.
Therefore when $x = A$, the acceleration $= - A\omega^2$

Functions of two or more variables

Let V be a function of x and y, ie $V = F(x,y)$

At point P $\quad V = F(x,y)$
At point P' $\quad V + \delta V = F(x + \delta x, y + \delta y)$
therefore

$$\begin{aligned} \delta V &= F(x + \delta x, y + \delta y) - F(x,y) \\ &= F(x + \delta x, y + \delta y) - F(x, y + \delta y) + F(x, y + \delta y) - F(x,y) \\ &= \delta F \text{ due to } \delta x \text{ with } y \text{ kept constant at } y + \delta y \\ &\quad + \delta F \text{ due to } \delta y \text{ with } x \text{ kept constant at } x \end{aligned}$$

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y$$

where $\frac{\partial F}{\partial x}$ means the differential of F with respect to x with y kept constant

and $\frac{\partial F}{\partial y}$ means the differential of F with respect to y with x kept constant

These are called the partial differentials of F (261)

Similarly, if V = F(x,y,z)

$$\delta V = \frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z \qquad (262)$$

Second Order Differentials with two variables

Let F be a function of x and y

By considering the small elements δx and δy it can be seen that

$$\frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial^2 F}{\partial y\,\partial x} \qquad (263)$$

16 AREAS AND VOLUMES

Areas and Volumes

The Area under a Curve can be found by Integrating an elemental strip

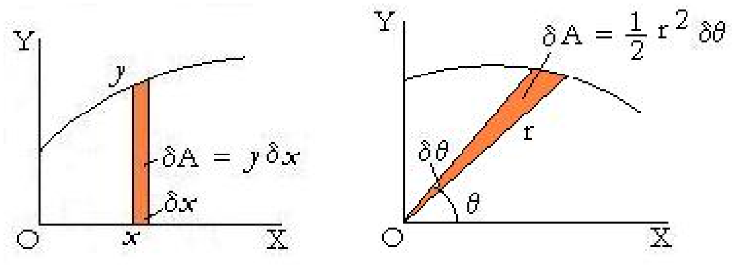


Figure 78: Area of elemental strips in Cartesian and in Polar Co-ordinates. (264)

Area of an Ellipse

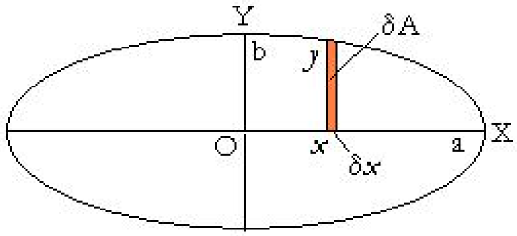


Figure 79: Area of an elipse

For an ellipse $x^2/a^2 + y^2/b^2 = 1$
$y = b\sqrt{[1 - (x / a)^2]} = (b/a)\sqrt{(a^2 - x^2)}$

Area = 4 times area of a quadrant
$= 4 \int y \, dx$ from $x = 0$ to $x = a$
$= 4 (b/a) \int \sqrt{(a^2 - x^2)} \, dx$

Put $x = a \sin\theta$ therefore $dx = a \cos\theta \, d\theta$
$\theta = 0$ when $x = 0$ and $\theta = \pi/2$ when $x = a$
Area $= 4 (b/a) \int \sqrt{(a^2 - a^2 \sin^2\theta)} \, a \cos\theta \, d\theta$
Area $= 4 a b \int \cos^2\theta \, d\theta = 4 a b \int [\cos(2\theta) + 1] \, d\theta$
$= 4 a b (½)[\sin(2\theta) + \theta]$ from $\theta = 0$ to $\theta = \pi/2$
$= 4 a b (½)[0 + \pi/2 - 0 - 0]$
$= \pi ab$ (265)

Volume of a Pyramids and Cones

Consider a pyramid with the base any shape and base area A.

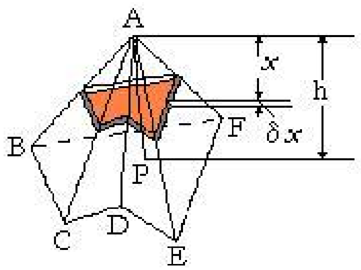


Figure 80: Pyramid any shape

Let the height of the apex be h perpendicular to the plane of the base.

Let a plate parallel to the base, distance x from the apex have thickness δx
The volume $\delta V = A\,(x/h)^2\,\delta x$
Hence the volume of the pyramid

$$V = \int_0^h [A(x^2/h^2]dx = (A/h^2)\,[x^3/3]_0^h = A\,h/3$$

Volume of a pyramid with base any shape = (1/3) (Base Area) (height) (266)

Volume of a Cone with circular base $V = \pi r^2 h/3$ (267)
Volume of a Pyramid with square base a by a $V = a^2 h/3$ (268)

Volume of a Tetrahedron

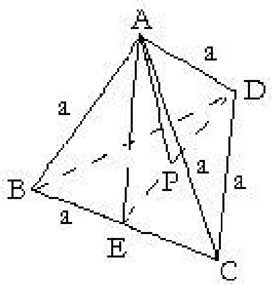


Figure 81: Tetrahedron

In the diagram, ABCD is a Tetrahedron with all sides length a

AP is perpendicular to the plane of BCD
AE and DE are perpendicular to BC

Area of the base = (1/2) BC DE
= $(1/2)\,a\,(\sqrt{3})2)\,a$
= $(\sqrt{3}/4)\,a^2$

Let the height AP = h

$PD = \sqrt{(a^2 - h^2)}$
$AE = (\sqrt{3}/2)\,a$ and $ED = (\sqrt{3}/2)\,a$

$PE = \sqrt{(AE^2 - h^2)} = \sqrt{\{(3/4)\,a^2 - h^2\}}$
But $PE = ED - PD$

Thus $\sqrt{[(3/4)\,a^2 - h^2]} = (\sqrt{3}/2)\,a - \sqrt{(a^2 - h^2)}$

Square both sides $(3/4)\,a^2 - h^2 = (3/4)\,a^2 - \sqrt{3}\,a\,\sqrt{(a^2 - h^2)} + (a^2 - h^2)$
Subtract $(3/4)\,a^2 - h^2$ from both sides $0 = -\sqrt{3}\,a\,\sqrt{(a^2 - h^2)} + a^2$
Divide by a and rearrange $\sqrt{3}\,\sqrt{(a^2 - h^2)} = a$
Square both sides $3\,(a^2 - h^2) = a^2$
Therefore $3h^2 = 2a^2$
Take square roots of both side;
$h = \sqrt{(2/3)}\,a$ ignoring the negative value

Volume of a Tetrahedron with side a
$V = (1/3) \times \text{Base area} \times \text{height} = 1/3 \times (½\,BC \times DE) \times h$
$= (1/3)\,(\sqrt{3})\,(1/4)\,a^2\,(\sqrt{2}/\sqrt{3})\,a = a^3/(6\sqrt{2})$ (269)

Volume of Revolution

The Volume of Revolution is the volume obtained by rotating the curve $y = f(x)$ about the X axis.

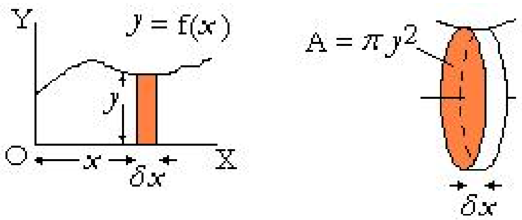


Figure 82: Volume of Revolution

Let V.= Volume of Revolution

Rotate an element of the curve about the X axis to obtain a disc
Area of the disc $A = \pi y^2$
Thickness of the disc = δx
Volume of Revolution of elemental disc $\delta V = \pi y^2 \delta x$

Hence the Volume of Revolution is $V = \int \pi y^2 \, dx$ (270)

Volume of a Sphere
The Curve is $x^2 + y^2 = a^2$ therefore $y^2 = (a^2 - x^2)$ from $-a$ to $+a$

$$V = \int_{-a}^{a} \pi y^2 \delta x = \int_{-a}^{a} \pi (a^2 - x^2) \delta x$$

$$V = [\pi (a^2 x - (1/3) x^3)]_{-a}^{a} = (4/3) \pi a^3$$

The Volume of a sphere is $4/3 \pi a^3$ (271)

The Sphere can be considered to be made up of many small pyramids each with height a and base area δA

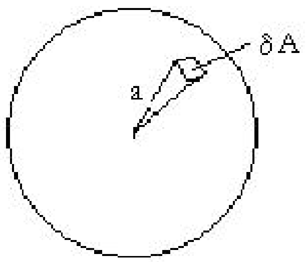


Figure 83: Element of volume of a sphere
The volume of this small pyramid $\delta V = (1/3)\, a\, \delta A$

Therefore the Volume of the whole sphere is $V = (1/3)\, a\, A$
where A is the total surface area of the sphere

But $V = (4/3)\,\pi\, a^3$ Therefore $A = 4\,\pi\, a^2$

Surface Area of a Sphere $= 4\,\pi\, a^2$ (272)

This is the same as the Curved Surface Area of a Cylinder that exactly fits over the Sphere.

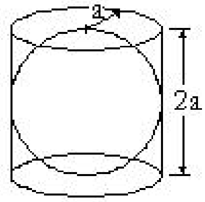


Figure 84: Surface Area of a sphere

Alternatively, the Surface Area can be obtained by rotating the elemental arc δs about the X axis

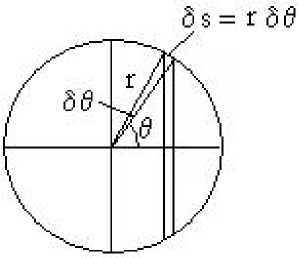


Figure 85: Element of surface area of a sphere

$r = a$
$y = r \text{ Sin } \theta = a \text{ Sin } \theta$
$\delta s = r\,\delta\theta = a\,\delta\theta$
$\delta A = 2\pi y = 2\pi a \text{ Sin } \theta\ \delta s$
$\quad = 2\pi a^2 \text{ Sin } \theta\ \ \delta\theta$

Integrate from 0 to π
$A = [-\,2\pi a^2 \text{ Cos } \theta]$ from 0 to π
$\quad = -\,2\,\pi a^2\,]\,[-\,1 - 1] = 4\,\pi\, a^2$

17 MAXIMA AND MINIMA

Maxima and minima where $y = f(x)$

The maximum and minimum values of a function can be found by the use of Calculus.

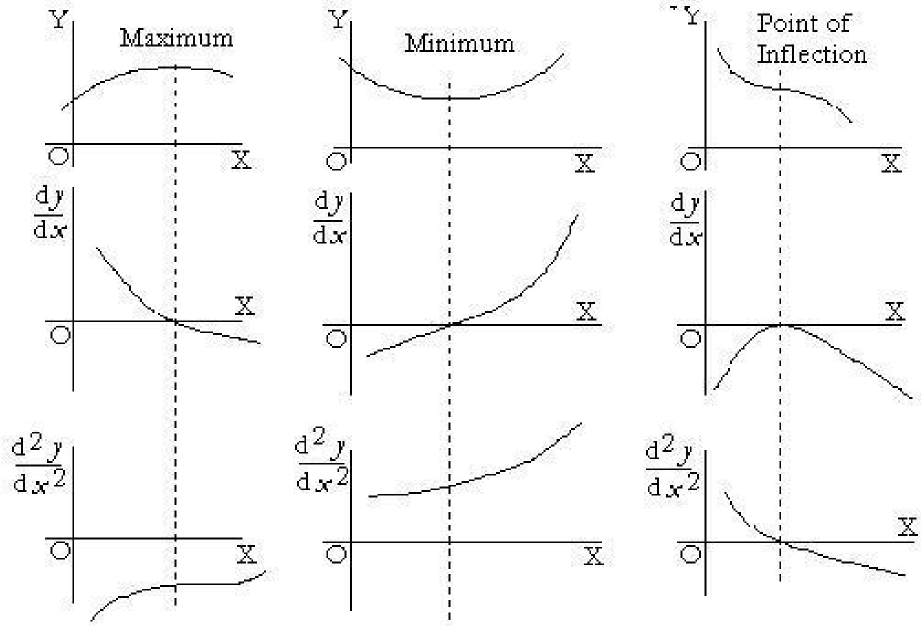


Figure 86: Maximum and Minimum

Let $y = f(x)$

Analysing the diagram, it will be seen that;

y is a maximum if $dy/dx = 0$ and d^2y/dx^2 is negative (273)

y is a minimum if $dy/dx = 0$ and d^2y/dx^2 is positive (274)

y is a point of inflection if $dy / dx = 0$ and d^2y / dx^2 is zero (275)

Example

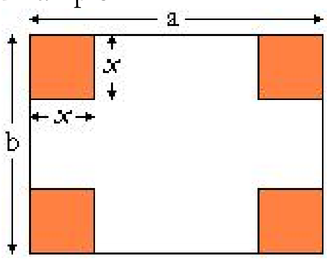


Figure 87: Maximum

An open water tank is to be made from a sheet of metal length a and width b

The shaded parts are to be cut out and the sides bent up.

Find the value of x for the tank to hold the maximum amount of water.

$V = (a - 2x)(b - 2x)\,x = abx - 2(a + b)\,x^2 + 4x^3$

$dV/dx = ab - 4(a + b)\,x + 12x^2$

$d^2V/dx^2 = -4(a + b) + 24x$

For a Maximum value of V, $dv/dx = 0$ and d^2V/dx^2 is negative

$dV/dx = 0$ when $12x^2 - 4(a + b)\,x + ab = 0$

ie when $x = [+ 4(a + b) \pm \sqrt{\{16(a + b)^2 - 48ab\}}\,] / 24$

$= [\,(a + b) \pm \sqrt{\{(a + b)^2 - 3ab\}}\,] / 6$

$= [\,(a + b) \pm \sqrt{(a^2 - ab + b^2)}\,] / 6$

The + ive sign gives a negative value of V, therefore, neglecting the value with the +ive sign;

$x = [\,(a + b) - \sqrt{(a^2 - ab + b^2)}\,] / 6$

At this value, $d^2V / dx^2 = -4(a + b) + 4(a + b) - 4/(a^2 - ab + b^2) = -4/[(a - b)^2 + ab]$ which is negative

Therefore V has a maximum value when $x = [(a + b) - \sqrt{(a^2 - ab + b^2)}] / 6$

Maxima and minima where $F = f(x,y)$

As above, $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ are conditions for a maximum or minimum point.

For a maximum point, $\partial^2 F/\partial x^2$ is negative and $\partial^2 F/\partial y^2$ is negative
For a minimum point, $\partial^2 F/\partial x^2$ is positive and $\partial^2 F/\partial y^2$ is positive

Although these conditions are necessary for maximum and minimum points, they are not enough without the additional condition. $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\,\partial y]^2$
see "Advanced Calculus" by A E Taylor or "Advanced Calculus" by Sokolnikoff.
If this condition is met, then $\partial^2 F/\partial x^2$ and $\partial^2 F/\partial y^2$ must both be the same sign.

Hence the conditions for a **maximum point** are;
$\partial F/\partial x = 0$ and $\partial F/\partial y = 0$
and $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\,\partial y]^2$ and $\partial^2 F/\partial x^2$ is negative (276)

And the conditions for a **minimum point** are;
$\partial F/\partial x = 0$ and $\partial F/\partial y = 0$
and $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] > [\partial^2 F/\partial x\,\partial y]^2$ and $\partial^2 F/\partial x^2$ is positive (277)

Saddle Point
If $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$
and $[\partial^2 F/\partial x^2]\,[\partial^2 F/\partial y^2] < [\partial^2 F/\partial x\,\partial y]^2$ then the point is a Saddle Point (278)

Example
A ring with centerline in the plane $x = y$,
has the ring thickness 2a and radius b where a<<b

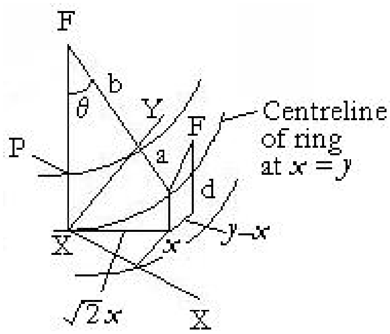


Figure 88: Saddle Point

At a point on the ring surface near Point P

$F = \sqrt{[a^2 - (y-x)^2]} + d$

Where $d = b(1 - \cos\theta)$ and $b \sin\theta = \sqrt{2}\,x$

θ is small, therefore $d = \frac{1}{2} b \theta^2 = x^2 / b$

$F = \sqrt{[a^2 - (y-x)^2]} + x^2 / b$

$$\partial F/\partial x = 1/[2\sqrt{\{a^2 - (y-x)^2\}}].[-2(y-x)(-1)] + 2x/b = (y-x)/[\sqrt{\{a^2 - (y-x)^2\}}] + 2x/b$$
$$\partial F/\partial y = -(y-x)/[\sqrt{\{a^2 - (y-x)^2\}}]$$

At $x = 0$ and $y = 0$, $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$
$\partial^2 F/\partial x^2 = -1 / [\sqrt{\{a^2 - (y - x)^2\}}] - (y - x)^2 / [a^2 - (y - x)^2]^{3/2} + 2/b$

at $x = 0$ and $y = 0$, $\partial^2 F/\partial x^2 = -1/a + 2/b$ which is negative if $a < ½ b$
Similarly, at $x = 0$ and $y = 0$, $\partial^2 F/\partial y^2 = -1/a$ which is negative but clearly point P is not a maximum

$\partial^2 F/\partial x\, \partial y = (y - x)^2/[a^2 - (y - x)^2]^{3/2} + 1 / [\sqrt{\{a^2 - (y - x)^2\}}]$
when $x = 0$ and $y = 0$, $\partial^2 F/\partial x\, \partial y = 1/a$

$[\partial^2 F/\partial x^2]\, [\partial^2 F/\partial y^2] - [\partial^2 F/\partial x\, \partial y]^2 = (-1/a)(-1/a + 2/b) - (1/a)^2 = -2b/a$
The additional condition for a maximum is not met and this shows that point P is a Saddle Point.

Numerical solution

For practical applications, it is easy to write a program that finds the maximum or minimum value of a function $f(x,y)$ by numerical analysis.

Values are assigned to x and y by nested "FOR TO" loops and the maximum or minimum value selected.

18 GRAPHS

Length of Arc

Length of an Arc in Cartesian Co-ordinates

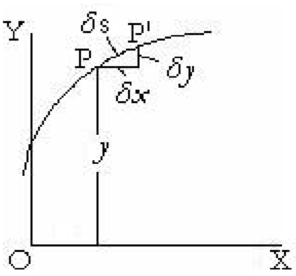


Figure 89: Length of an Arc

$\delta s^2 = \delta x^2 + \delta y^2$
$\delta s = \sqrt{(\delta x^2 + \delta y^2)}$
$\delta s = \delta x \sqrt{[1 + (\delta y/\delta x)^2]}$

$$s = \int \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dx \qquad (279)$$

Length of an arc in Polar Co-ordinates

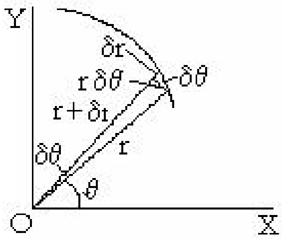


Figure 90: Length of Arc in Polar Co-ordinates

$\delta s^2 = \delta r^2 + (r\,\delta\theta)^2$
$\delta s = \sqrt{(\delta r^2 + (r^2\,\delta\theta^2)}$
$= \sqrt{[(\delta r/\delta\theta)^2 + r^2]}\,\delta\theta$
Integrate
$s = \int \sqrt{[(dr/d\theta)^2 + r^2]}\,d\theta$
$s = \int [\sqrt{[r^2 + (dr/d\theta\_)^2]}\,d\theta$ (280)

Example

Find the length of a catenary chain given by $y = c\,[\text{Cosh}\,(x/c) - 1]$ between $x = 0$ and $x = a$

$y = c\,[\text{Cosh}\,(x/c) - 1]$

$dy/dx = c\,[\text{Sinh}\,(x/c)]\,(1/c) = \text{Sinh}\,(x/c)$

$s = \int \sqrt{[\,1 + \text{Sinh}^2\,(x/c)\,]}\,dx$

$s = \int \sqrt{[\text{Cosh}^2\,(x/c)\,]}\,dx = \int \text{Cosh}\,(x/c)\,dx = c\,\text{Sinh}\,(x/c)$ from $x = 0$ to $x = a$

$s = c\,\text{Sinh}\,(a/c)$

Radius of Curvature

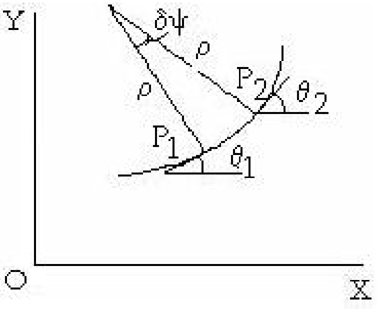


Figure 91: Radius of Curvature

Let the radius of curvature be ρ

$\tan\ \theta_1 \ = \ dy/dx$ at P1

$\tan\ \theta_2 \ = \ dy/dx$ at P2 $= \ dy/dx + \delta(dy/dx)$

$\delta\psi \ = \ \theta_2 \ - \ \theta_1$

$\tan \delta\psi \ = \ (\tan \theta_2 - \tan \theta_1) \ / \ (1 + \tan \theta_1 \tan \theta_2)$

$\qquad = \ \delta(dy/dx) \ / \ [1 \ + \ (dy/dx) \ \{ dy/dx + \delta(dy/dx) \} \ ]$

$\delta\psi$ is small therefore $\delta\psi \ = \tan \delta\psi$

$d\psi/dx \ = $ Limit as δx tends to zero $[\ \delta\psi/\delta x \ ]$

$\qquad =$ Limit as δx tends to zero $[\ (\tan \delta\psi) \ / \ \delta x \ ]$

$\qquad =$ Limit as δx tends to zero $[\ \delta(dy/dx) \ / \ \{1 + (dy/dx) \ \{dy/dx + \delta \ (dy/dx) \ \} \ ] \ / \ \delta x$

$\qquad = \ d^2y/dx^2 \ / \ [1 + (\ dy/dx)^2 \ ]$

But $\rho \ \delta\psi \ = \ \delta s$

Therefore $(\rho \ \delta\psi \ )^2 \ = \ \delta s^2 \ = \ \delta x^2 \ + \ \delta y^2$

$\qquad \rho^2 \ (\ d\psi/dx)^2 = 1 + \ (dy/dx)^2$

Therefore

$\qquad \rho^2 = \{1 + \ (dy/dx)^2 \ \} \ / \ (\ d\psi/dx)^2$

$\qquad\quad = \{1 \ + \ (dy/dx)^2 \ \} \ \{ \ 1 + (\ dy/dx)^2 \ \}^2 \ / \ \{d^2y/dx^2 \ \}^2$

$\qquad\quad = \{ \ 1 + (\ dy/dx)^2 \ \}^3 \ / \ \{ \ d^2y \ / \ dx^2 \ \}^2$

$$\rho \ = \ \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} \qquad (281)$$

Example
Find the radius of curvature of an ellipse $x^2/a^2 + y^2/b^2 = 1$ at (0, b) and at (a , 0)

Differentiate with respect to x
$2x/a^2 + (2y/b^2)\, dy/dx = 0$
$dy/dx = -(b^2/a^2)(x/y)$

$d^2y/dx^2 = -(b^2/a^2)[1/y - (x/y^2)dy/dx]$
$= -(b^2/a^2)(1/y)[1 + (b^2/a^2)(x^2/y^2)$
$= -(b^4/a^2)(1/y^3)(y^2/b^2 + x^2/a^2) = -b^4/(a^2y^3)$ since $x^2/a^2 + y^2/b^2 = 1$

$\rho = \{1 + (dy/dx)^2\}^{3/2} / d^2y/dx^2$
$= [1 + (b^4/a^4)(x^2/y^2)]^{3/2}/[-b^4/(a^2y^3)]$
$= -[a^4y^2 + b^4x^2]^{3/2}/[(a^6y^3)(b^4/(a^2y^3)]$
$= -[a^4y^2 + b^4x^2]^{3/2}/(a^4b^4)$

The negative sign means the centre of curvature is below the curve and can be ignored

$\rho = [a^4y^2 + b^4x^2]^{3/2}/(a^4b^4)$

At point (0 , b) $\rho = [a^4b^2 + b^40^2]^{3/2} / (a^4b^4) = a^2/b$

At point (a , 0) $\rho = [a^40^2 + b^4a^2]^{3/2} / (a^4b^4) = b^2/a$

Tangent to a Curve

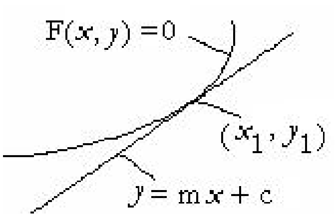


Figure 92: Tangent to curve $F(x,y) = 0$

Let $y = mx + c$ be a Tangent to the Curve $F(x,y) = 0$
with the point of contact at (x_1, y_1)

Differentiate $F(x,y) = 0$ and rearrange to get $dy/dx = F_1(x, y)$

The Tangent is the line through point (x_1, y_1) with slope $F_1(x, y)$

The equation for the Tangent at point (x_1, y_1) is therefore

$y - y_1 = [F_1(x, y)]\,(x - x_1)$

Example
The Tangents to an ellipse $x^2/a^2 + y^2/b^2 = 1$ pass through an external point (x_2 , y_2)
Find the values of x_1 and y_1 for the points of contact with the elipse

Differentiate $2x/a^2 + (2y/b^2)\, dy/dx = 0$ therefore $dy/dx = -(b^2/a^2)(x/y)$
Tangents are $(y - y_2) = -(x - x_2)(b^2/a^2)(x_1 / y_1)$
Where $x_1^2/a^2 + y_1^2/b^2 = 1$

Therefore $x_1^2 = (a/b)^2 (b^2 - y_1^2)$ and $x_1 = \pm(a/b)\sqrt{(b^2 - y_1^2)}$
Substituting for x_1 and x_1^2 in the equation for the Tangents and simplifying
$[a^2 (a^2 y_2^2 + b^2 x_2^2)]\, y_1^2 - [2a^4 b^2 y_2]\, y_1 + a^4 b^4 - a^2 b^4 x_2^2 = 0$
This is a quadratic in y_1 and the solution after simplifying is
$y_1 = b^2 [(a^2 y_2 \pm x_2 \sqrt{(a^2 y_2^2 - a^2 b^2 + b^2 x_2^2)})] / (a^2 y_2^2 + b^2 x_2^2)$
$x_1 = (a/b) \sqrt{(b^2 - y_1^2)}$

19 VECTORS

Definitions of Scalar and Vector quantities

A Scalar quantity has Magnitude, for example a Number.

A Vector has both Magnitude and Direction, for example Velocity. You cannot say where you will be when you have flown for one hour from London at 500 miles per hour unless you also know the direction you have flown.

Thus to completely define velocity or acceleration or force or many other quantities, it is also necessary to define the direction. This is done by defining the quantity as a Vector. Note that the Vector definition does not also include the position of the quantity, only the direction.

Addition of Vectors

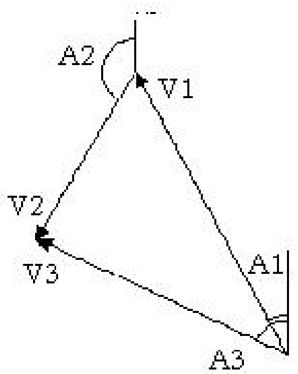


Figure 93: Addition of Vectors

Suppose an aircraft is moving with Velocity V1 relative to the air, the wind is blowing with Velocity V2 and the aircraft is moving relative to the ground with Velocity V3.

Then the Vector **V3** = the sum of Vectors **V1** and **V2**

Vector addition;

$$\mathbf{V3} = \mathbf{V1} + \mathbf{V2} \tag{282}$$

If Vector V1 is in a direction at an angle A1 to a fixed direction , say anticlockwise from North.

Similarly Vectors V2 and V3 are in directions at an angle A2 and A3 respectively to this fixed direction.

Let V1, V2 and V3 be the scalar magnitudes of **V1**, **V2** and **V3**

Then V3 Cos A3 = V1 Cos A1 + V2 Cos A2
And V3 Sin A3 = V1 Sin A1 + V2 Sin A2

Thus the single Vector equation **V3** = **V1** + **V2** is a shorthand way of writing the two Scalar equations;
V3 = √[(V1 Cos A1 + V2 Cos A2)^2 + (V1 Sin A1 + V2 Sin A2)^2]
A3 = Arc Tan [(V1 Sin A1 + V2 Sin A2) /(V1 Cos A1 + V2 Cos A2)] (283)

Operator j

The Operator J followed by a Vector means the Vector's direction is rotated by 90^0 anticlockwise.

The Operator j is assumed to follow the usual rules of Algebra except that the order of j and the Vector cannot be changed.

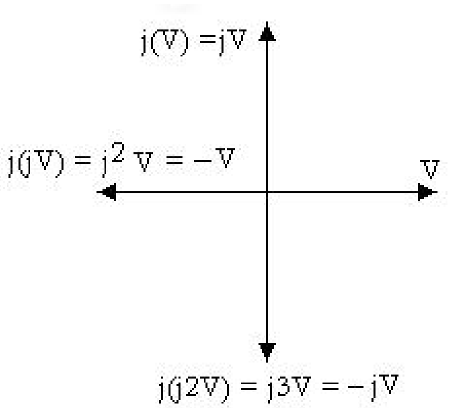


Figure 94: Operator j

Operating with j twice on the Vector V gives a vector in exactly the opposite direction to V and of the same Magnitude.

Thus j^2 **V** = − **V** (284)
And j^3 **V** = − j **V** (285)

Thus j^2 operating on a Vector reverses the direction of the Vector or is equivalent to multiplying the Vector by (-1)

Operator h

The operator h rotates the Vector by 120^0
Therefore **V** + h**V** + h^2**V** = 0
h**V** = (- 0.5 + √3/2 j)**V**

Vector in three Dimensions

A Vectors in three dimensions can be defined in terms of its components in three directions mutually at right angles

Unit Vectors (ie vectors with unit length) in directions Ox, Oy and Oz are called **i**, **j** and **k**

The right hand convention for the relative directions

Hold out the right hand as if to shake hands.
Going from the tips of the fingers towards the elbow, the fingers point in the direction of **i**, the palm faces in the direction of **j** and the thumb points in the direction of **k**.

Vectors **i**, **j** and **k** have unit length in directions **i**, **j** and **k**

.

Thus any vector **V** can be defined as $\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$.
where V_x, V_y and V_z are the scalar magnitudes of each component

Let OA be vector V with components V_x, V_y and V_z

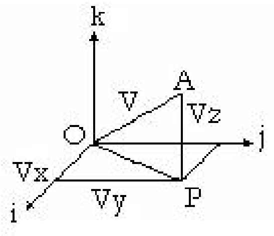


Figure 95: Components of a Vector

$OP^2 = V_x^2 + V_y^2$
$V^2 = OA^2 = OP^2 + V_z^2$

Hence $V = \sqrt{[V_x^2 + V_y^2 + V_z^2]}$ (286)

A Matrix is a convenient way of defining a system of vectors
Vectors **A, B, C** and **D** are defined by the Matrices

$$\begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \\ d_i & d_j & d_k \end{vmatrix} \begin{vmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{vmatrix} \quad \text{or by} \quad \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \\ d_i & d_j & d_k \end{vmatrix} \qquad (287)$$

These are a shorthand way of writing

$\mathbf{A} = a_i\,\mathbf{i} + a_j\,\mathbf{j} + a_k\,\mathbf{k}$
$\mathbf{B} = b_i\,\mathbf{i} + b_j\,\mathbf{j} + b_k\,\mathbf{k}$
$\mathbf{C} = c_i\,\mathbf{i} + c_j\,\mathbf{j} + c_k\,\mathbf{k}$
$\mathbf{D} = d_i\,\mathbf{i} + d_j\,\mathbf{j} + d_k\,\mathbf{k}$

where a_i is the magnitude of the component of **A** along the **i** axis etc

Scalar or Dot Product of Vectors

If two vectors **U** and **V** have an angle θ between them, then the Vector Dot Product is a shorthand way of writing their product resolved in the same direction as one of them.

$\mathbf{V \bullet U} = V\,U\,\mathrm{Cos}\,\theta$ (288)

where V and U are the magnitudes of the vectors and θ is the angle between them
The Vector Dot Product is a scalar quantity.

Thus $\mathbf{i\bullet i} = \mathbf{j\bullet j} = \mathbf{k\bullet k} = 1$ and $\mathbf{i\bullet j} = \mathbf{j\bullet k} = \mathbf{k\bullet i} = 0$ (289)

If $\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$
and $\mathbf{U} = U_x\,\mathbf{i} + U_y\,\mathbf{j} + U_z\,\mathbf{k}$
Then $\mathbf{V{\bullet}U} = V_x U_x\,\mathbf{i{\bullet}i} + V_x U_y\,\mathbf{i{\bullet}j} + V_x U_z\,\mathbf{i{\bullet}k}$
$+ V_y U_x\,\mathbf{j{\bullet}i} + V_y U_y\,\mathbf{j{\bullet}j} + V_y U_z\,\mathbf{j{\bullet}k}$
$+ V_z U_x\,\mathbf{k{\bullet}i} + V_z U_y\,\mathbf{k{\bullet}j} + V_z U_z\,\mathbf{k{\bullet}k}$
$\mathbf{V{\bullet}U} = V_x U_x + V_y U_y + V_z U_z$ (290)

Angle between Vectors

From (288) and (290) $\quad V\,U\,\mathrm{Cos}\,\theta = V_x U_x + V_y U_y + V_z U_z$

From (288), the angle between vectors **V** and **U** is given by

$$\mathrm{Cos}\,\theta = \frac{Vx\,Ux + Vy\,Uy + Vz\,Uz}{\sqrt{[(Vx^2+Vy^2+Vz^2)(Ux^2+Uy^2+Uz^2)]}} \qquad (291)$$

The Vectors are at right angles when
$V_xU_x + V_yU_y + V_zU_z = 0$ (292)

Direction Cosines

The cosines of the angles between a vector **V** and each of the axes are called the Direction Cosines. These define the direction of **V**.

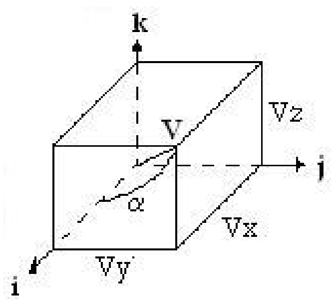


Figure 96: Direction Cosines

Let **v** be a unit vector parallel to the vector **V**
Then $\mathrm{Cos}\,\alpha = \mathbf{v{\bullet}i}$, $\quad \mathrm{Cos}\,\beta = \mathbf{v{\bullet}j}$ and $\mathrm{Cos}\,\gamma = \mathbf{v{\bullet}k}$

If Ux, Uy and Uz are the components of unit vector **v** along each axis
then $Ux^2 + Uy^2 + Uz^2 = 1^2$
Also $Ux = \mathrm{Cos}\,\alpha$, $Uy = \mathrm{Cos}\,\beta$ and $Uz = \mathrm{Cos}\,\gamma$
Therefore $\mathrm{Cos}^2\alpha + \mathrm{Cos}^2\beta + \mathrm{Cos}^2\gamma = 1$ (293)

Vector or Cross Product of Vectors

The Vector Cross Product of two vectors is a Vector with direction perpendicular to the plane of the two vectors. The magnitude is proportional to the Sine of the angle between them.

$$\mathbf{V}\ \mathbf{X}\ \mathbf{U} = V\ U\ \mathrm{Sin}\ \theta\ \mathbf{a} \tag{294}$$

where V and U are the magnitudes of the vectors

θ is the angle between them

and **a** is a unit vector perpendicular to the plane of **V** and **U**

The Vector Cross Product is a Vector and $\mathbf{V}\ \mathbf{X}\ \mathbf{U} = -\ \mathbf{U}\ \mathbf{X}\ \mathbf{V}$ (295)

By Convention, the product is positive in the direction of a corkscrew turned clockwise from the first vector to the second vector.

Thus $\mathbf{i}\ \mathbf{X}\ \mathbf{j} = \mathbf{k}$, $\quad \mathbf{j}\ \mathbf{X}\ \mathbf{k} = \mathbf{i}$ $\quad$ and $\quad \mathbf{k}\ \mathbf{X}\ \mathbf{i} = \mathbf{j}$ (296)

The product is positive when the vectors fall in the sequence **i j k i j k**

It follows that when the vectors are in the reverse of this sequence, the product is negative

Let $\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$ and $\mathbf{U} = U_x\,\mathbf{i} + U_y\,\mathbf{j} + U_z\,\mathbf{k}$

$$\mathbf{V\,X\,U} = (V_yU_z - V_zU_y)\,\mathbf{i} + (V_zU_x - V_xU_z)\,\mathbf{j} + (V_xU_y - V_yU_x)\,\mathbf{k} \tag{297}$$

This result can be expressed in Determinant form

$$\mathbf{V\,X\,U} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ V_x & V_y & V_z \\ U_x & U_y & U_z \end{vmatrix} \tag{298}$$

Scalar Triple Product of Vectors

Let Vectors **A**, **B** and **C** define three adjacent edges of a parallelepiped

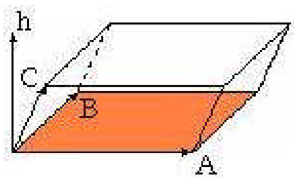


Figure 97: Scalar Triple Product

A X B = Area of Base X Unit Vector **h**

But **h•C** is the height between the Base and top
Thus the Volume of the parallelepiped = **A X B•C**
This is the Triple Vector Product and is a Scalar quantity

By Symmetry, Volume = **A X B•C** = **B X C•A** = **C X A•B**
Each function has the same sequence and the Cross Product must be evaluated first.

Grad

If a scalar quantity (eg temperature or pressure) is defined at any point in a three dimensional volume, then there is a surface linking all adjacent points with the same value. The **grad** or gradient of the quantity, at any point on this surface, is a vector normal to the surface with magnitude equal to the rate of change of the quantity in this direction. Let the scalar quantity be V at point P co-ordinates (x, y, z)

Then at point P
grad V = dV/dn **n**
where **n** is the unit vector normal to the surface and dV/dn is the rate of change of V in direction **n**
grad V is not dependent on the axes

grad V is sometimes written $\boldsymbol{\Delta}$V, thus $\boldsymbol{\Delta} V = (\partial V/\partial x)\,\mathbf{i} + (\partial V/\partial y)\,\mathbf{j} + (\partial V/\partial z)\,\mathbf{k}$

ie **grad** is equivalent to the operator $\boldsymbol{\Delta} = (\partial/\partial x)\,\mathbf{i} + (\partial\,/\partial y)\,\mathbf{j} + (\partial\,/\partial z)\,\mathbf{k}$ (299)

Differentiation of a Vector

A Vector quantity (eg velocity of a fluid, or electric field) can be defined at any point P (x,y,z) in a three dimensional volume.

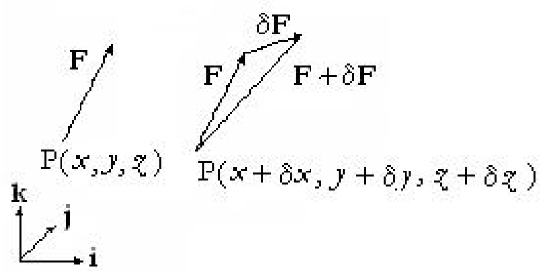


Figure 98: Differentiating a Vector

Let the Vector quantity be $\mathbf{F}$ at point (x ,y, z)
The Vector quantity is $\mathbf{F} + \delta\mathbf{F}$ at (x+ δx), (y + δy), (z + δz)
where $\delta\mathbf{F} = (\partial\mathbf{F}/\partial x)\,\delta x + (\partial\mathbf{F}/\partial y)\,\delta y + (\partial\mathbf{F}/\partial z)\,\delta z$

The component of $\delta\mathbf{F}/\delta x$ along the X axis = $(\partial\mathbf{F}/\partial x)\bullet\mathbf{i}$ (300)

Divergence and Curl of a Vector quantity

Div $\mathbf{F}$ is defined as $(\partial\mathbf{F}/\partial x)\bullet\mathbf{i} + (\partial\mathbf{F}/\partial y)\bullet\mathbf{j} + (\partial\mathbf{F}/\partial z)\bullet\mathbf{k} = \boldsymbol{\Delta}\bullet\mathbf{F}$ (301)

and Curl $\mathbf{F}$ is defined as $(\partial\mathbf{F}/\partial x)\,\mathbf{X}\,\mathbf{i} + (\partial\mathbf{F}/\partial y)\,\mathbf{X}\,\mathbf{j} + (\partial\mathbf{F}/\partial z)\,\mathbf{X}\,\mathbf{k} = \boldsymbol{\Delta}\,\mathbf{X}\,\mathbf{F}$ (302)

Thus Div $\mathbf{F}$ is a scalar quantity and Curl $\mathbf{F}$ is a vector
Div $\mathbf{F}$ and Curl $\mathbf{F}$ are not dependent on the choice of the axes.

If $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$
Then Div $\mathbf{F} = \partial F_1/\partial x + \partial F_2/\partial y + \partial F_3/\partial z$
And Curl $\mathbf{F} = (\partial F_3/\partial y - \partial F_2/\partial z)\,\mathbf{i} + (\partial F_1/\partial z - \partial F_3/\partial x)\,\mathbf{j} + (\partial F_2/\partial x - \partial F_1/\partial y)\,\mathbf{k}$

Hence Curl $\mathbf{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}$

A Straight Line through two points in a three dimensional space

Let the points A and B be represented by Vectors **A** and **B** relative to a point O

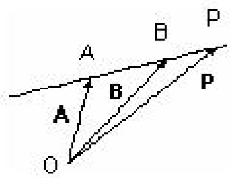


Figure 99: Line through two points

Then the line AB is the vector (**B** − **A**)
and the line AP is the vector (**P** − **A**)
But vector (**P** − **A**) = k (**B** − **A**)

Thus any point P on the line can
be represented by the vector **P** where

$$\mathbf{P} = \mathbf{A} + k(\mathbf{B} - \mathbf{A})$$

A Plane in a three dimensional space

Let ON be the line normal to the plane from a reference point O
Let **n** be the unit vector in direction ON
Let any point P on the plane be represented by the vector **P** reference to the point O

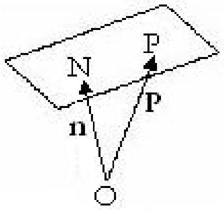


Figure 100: Vectors defining a plane

Then **P • n** = length ON
Thus the Vector Equation for the plane is **P • n** = N
where N is the length ON

The angle between the two planes

Let the direction of the normals to two planes be defined by unit vectors **n** and **m**.

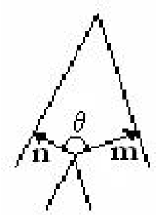


Figure 101: The angle between two planes

Let θ be the angle between **n** and **m**

Angle between the planes $= 360^0 - 90^0 - 90^0 - \theta$
$= 180^0 - \theta$ where θ is the angle between **n** and **m**

Let the components of **n** along axes 0x, 0y and 0z be Nx, Ny and Nz and the components of **m** along the same axes be Mx, My and Mz then the angle between **n** and **m** is given by equation (291) where **n** and **m** are unit vectors

Therefore $\cos\theta = Nx\,Mx + Ny\,My + Nz\,Mz$ (303)

This relation can be used to find the angle between two flat surfaces.

<u>Example</u>

The Gully G between Roof A and Roof B is to be prefabricated.

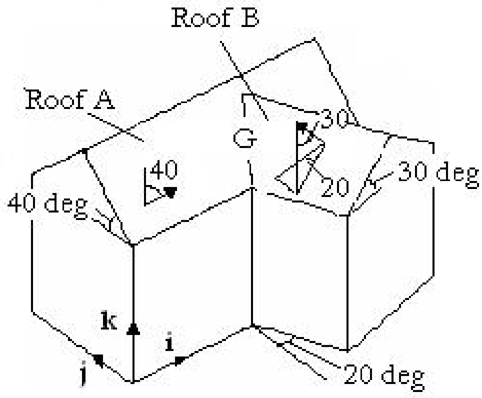


Figure 102: Angle between roofs

Calculate the angle of the Gully
Choose axes i, j and k as shown

Let **a** be the unit vector outwards and normal to Roof A
And **b** be the unit vector outwards and normal to Roof B

Put **a** and **b** in terms of their components along axes **i**, **j** and **k**
$\mathbf{a} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$
$\mathbf{b} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$

By inspection,
$\mathbf{a} = 0\,\mathbf{i} - \sin 40\,\mathbf{j} + \text{Cos } 40\,\mathbf{k}$
$\mathbf{b} = -\text{Sin } 30 \text{ Cos } 20\,\mathbf{i} - \text{Sin } 30 \text{ Sin } 20\,\mathbf{j} + \text{Cos } 30\,\mathbf{k}$

From (298)
Angle of the Gully = = $180^0 - \theta$
where $\text{Cos } \theta = Ax\,Bx + Ay\,By + Az\,Bz$
$= 0 + \text{Sin } 40 \text{ Sin } 30 \text{ Sin } 20 + \text{Cos } 40 \text{ Cos } 30$
$= 0.1099 + 0.6634 = 0.7733$
$\theta = 39$ degrees
Angle of the Gully = 180 – 39 =141 degrees

20 ARGAND DIAGRAM

Complex Numbers

The concept of Real and Complex numbers was mentioned briefly in Chapter 2 Algebra.

Complex numbers are of the form $A + i\,B$ where A and B are Real Numbers (positive or negative).
Complex Numbers obey the normal rules of Algebra with the additional rule that $i^2 = -1$

Let $A + i\,B = C + i\,D$
Then $(A - C)^2 = -(B - D)^2$
Therefore $A = C$ and $B = D$

Real and Complex axes

The Operator j^2 acting on any Vector has the effect of multiplying the Vector by (-1).
Thus the effect is equivalent to multiplying the vector by the scalar quantity $(i)^2$
Thus a convenient way to show a Complex Number $A + i\,B$ is to show it as Vectors **A** and j**B**

Thus if a diagram is drawn with the axis to the right called the Real axis, and the axis in a direction 90^0 anti-clockwise called the Complex axis, then any Complex Number can be represented by a point on the diagram. This diagram is called the Argand Diagram

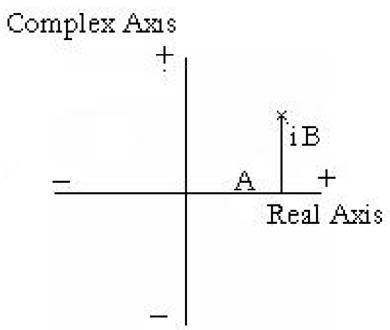


Figure 103: Argand Diagram

The Diagram shows the Complex Number A + i B

The Argand Diagram

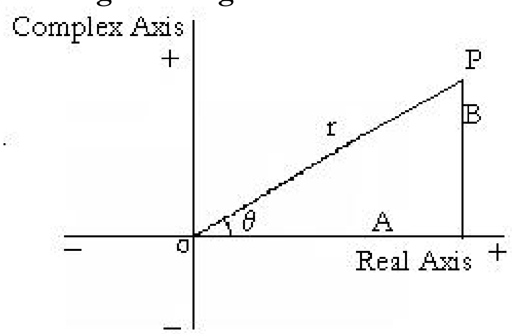


Figure 104: Modulus and Argument

P represents the Complex Number A + i B
$A = r \text{ Cos } \theta$ and $B = r \text{ Sin } \theta$
$r = + \sqrt{(A^2 + B^2)}$ is called the Modulus
$\theta = \text{Arc Tan } (B/A)$ is called the Argument

The Modulus is unique, but the Argument can have $2\pi n$ added or subtracted, where n is any whole number. The value between $-\pi$ and $+\pi$ is called the principle value.

On the Argand Diagram the Complex Number A + i B is

$$A + iB = r \text{ Cos } \theta + i \, r \text{ Sin } \theta$$
$$= r(\text{Cos } \theta + i \text{ Sin } \theta)$$

Therefore from (185) $A + iB = r e^{i\theta}$ (304)

Thus any Complex number A + i B can be represented by $r e^{i\theta}$
where $r = \sqrt{(A^2 + B^2)}$ and $\theta = \text{Arc Tan } (B/A)$

Sum of Complex Numbers

$(A + i B) + (C + i D) = (A + C) + i (B + D)$ (305)

Product of Complex Numbers

$(A + i B) (C + i D) = AC - BD + i (AD + BC)$ (306)
or $(X e^{i x}) (Y e^{i y}) = X Y e^{i (x + y)}$

n th power of a Complex Number

$(A + i B)^n = (r e^{i\theta})^n = r^n e^{i n \theta} = (r^n \text{Cos } n\theta) + i (r^n \text{Sin } n\theta)$ (307)
where $r = \sqrt{(A^2 + B^2)}$ and $\theta = \text{Arc Tan } (B / A)$

Quotient of a Complex Number

$(A + i B)/(C + i D) = (A + i B) (C - i D)/\{ (C + i D) (C - i D) \}$
$= [(AC + BD)/(C^2 + D^2)] + i [(BC - AD)/(C^2 + D^2)]$ (308)
or $(X e^{i x}) / (Y e^{i y}) = (X/Y) e^{i (x - y)}$ (309)

n th root of a Complex Number

Let $x^n = A + i B$
Find $x = \sqrt[n]{(A + i B)}$
Put $A + i B = R e^{i (\psi + 2k\pi)}$
where $R = \sqrt{(A^2 + B^2)}$ and $\psi = \text{Arc Tan } (B/A)$ and k is any integer

Let $x = r e^{i\theta}$
Then $r = \sqrt[n]{R}$ (310)
and $n\theta = 2k\pi + \psi$
$\theta = 2k\pi/n + \psi /n$ (311)
Put $k = 0, 1, 2, 3, \ldots\ldots (n - 1)$ to obtain n different values of x (312)

Example
Find the 8 solutions to the equation $x^8 = -1$
But $-1 = 1 e^{i (2k + 1) \pi}$
Let $x = r e^{i\theta}$ Therefore $r^8 = 1$ or $r = 1$
And $\theta = (2k + 1)\pi/8$
$\theta = \pi/8, 3\pi/8, 5\pi/8, \ldots\ldots\ldots\ldots 15\pi/8$

Therefore the 8 solutions are;
(0.92 + i 0.38), (0.38 + i 0.92), (− 0.38 + i 0.92), (− 0.92 + i 0.38), (− 0.92 − i 0.38),
(− 0.38 − i 0.92), (0.38 − i 0.92), and (0.92 − i 0.38)

21 DIFFERENTIAL EQUATIONS

Definitions

- Ordinary or Partial ie one or more variables
- Order is the highest order of derivative ie $d^n y / dx^n$ is order n
- Degree is the index (ie power) of the highest derivative when rationalised
- Complete Primitive. Solution with all arbitrary constants
- Particular Integral is any one solution derived from the Complete Primitive
- Singular Solution is a solution that cannot be derived in this way

A differential equation can be represented by a family of curves. The solution of a differential equation of n th order contains n arbitrary constants.

To sketch $dy/dx = f(x, y)$
If dy/dx is finite for all finite values of x and y, the family of curves may be sketched by considering dy/dx and d^2y/dx^2. One curve and one curve only passes through every point on the plane.

A Linear Differential Equation is of the form;

$$F_n\,(d^n y/dx^n) + F_{n-1}\,(d^{n-1}y/dx^{n-1}) + ... + F_1\,(dy/dx) + F_0(y) + f(x) = 0 \qquad (313)$$

Solution of Differential Equations by Substitution

Solution of a Linear Differential Equation
where $F_n(x)$ etc are any finite series of x

Put $y = a_0 + a_1 x + a_2 x^2/2! + a_3 x^3/3! + \ldots. + a_r x^r/r! + \ldots$ (314)

Substitute in the differential equation and equate coefficients
The method fails if the solution cannot be expressed in this form.

Warning. This method may produce an answer without the required number of arbitrary constants, eg a particular case where an arbitrary constant is zero.

Example

$$d^3y / dx^3 = dy / dx$$

Substituting;

$$a_3 + a_4 x + a_5 x^2 / 2! + a_6 x^3 / 3! + \ldots.. + \text{etc}$$
$$= a_1 + a_2 x + a_3 x^2 / 2! + a_4 x^3 / 3! + \ldots. + \text{etc}$$

Equating Coefficients;

$$a_1 = a_3 = a_5 = a_7 = \text{etc}$$
$$a_2 = a_4 = a_6 = \text{etc}$$

Hence $y = a_0 + a_1 (x + x^3/3! + x^5/5! + \ldots) + a_2 (x^2/2! + x^4/4! + x^6 / 6! \ldots)$
$= a_0 + a_1 \text{Sinh}\, x + a_2 \text{Cosh}\, x - a_2$
Put $a = (a_1 + a_2) / 2$, $b = (a_2 - a_1) / 2$ and $c = a_0 - a_2$
Then from (191) and (192) $y = a e^x + b e^{-x} + c$

Example

$$x\, d^2y/dx^2 - dy/dx - 6 x^2 - 7 = 0$$

Substituting;

$$x (a_2 + a_3 x + a_4 x^2/2! + a_5 x^3/3! + \ldots. + a_r x^{r-2}/(r - 2)! + \ldots$$
$$- (a_1 + a_2 x + a_3 x^2/2! + a_4 x^3/3! + \ldots. + a_r x^{r-1} /(r - 1)! + \ldots - 6x^2 - 7 = 0$$

Equating Coefficients;

$- a_1 - 7 = 0$ therefore $a_1 = - 7$
$a_2 - a_2 = 0$ therefore a_2 is indeterminate
$a_3 - (1/2) a_3 - 6 = 0$ therefore $a_3 = 12$
$a_4 / 2 - a_4 / 6 = 0$ therefore $a_4 = 0$
$a_r / (r - 2)! - a_r / (r - 1)! = 0$ therefore $a_r = 0$

The Solution is;

$$y = A - 7 x + B x^2 + (12 / 6) x^3$$
$$= A - 7 x + B x^2 + 2 x^3$$

Example

Show the method fails for $dy/dx = 1/x$

as Log x cannot be expanded by Maclaurim's Theorem

$y = a_0 + a_1 x + a_2 x^2/2! + a_3 x^3/3! +$

$x\, dy/dx = a_1 x + a_2 x^2 + a_3 x^3/2! + a_4 x^4/3! + ... = 1$

Therefore $a_1 = a_2 = a_3 = a_4 =$ ......etc $= 0$

There is no term to equate with 1

Exact Equations (First Order)

$M\, dx + N\, dy = 0$

and where M and N are functions of x and y and $\partial N/\partial x = \partial M/\partial y$ (315)

This is an exact equation and can be integrated at once.

Proof Let f be any function of x and y

$df = (\partial f / \partial x)\, dx + (\partial f / \partial y)\, dy$

Put $M = \partial f / \partial x$ and $N = \partial f / \partial y$

$df = M\, dx + N\, dy$

This equation can be integrated at once

$\partial M / \partial y = \partial^2 f / (\partial x\, (\partial y) = \partial^2 f / (\partial y\, (\partial x) = \partial N / \partial x$

Therefore if $\partial N/\partial x = \partial M/\partial y$, then the equation can be integrated at once.

Example (i)

$(12x + 5y - 9)\, dx + (5x + 2y - 4)\, dy = 0$

$M = (12x + 5y - 9)$

$N = (5x + 2y - 4)$

$\partial M/\partial y = 5$ and $\partial N/\partial x = 5$ Therefore the equation is Exact

Integrating

$6x^2 - 9x + \int 5y\, dx + \int 5x\, dy + y^2 - 4y + C = 0$

$6x^2 + 5xy + y^2 - 9x - 4y + C = 0$

Example (ii)

$\{\text{Cos}\, x\, \text{Tan}\, y + \text{Cos}\,(x + y)\}\, dx + \{\text{Sin}\, x\, \text{Sec}^2 y + \text{Cos}\,(x + y)\}\, dy = 0$

$\partial M/\partial y = \text{Cos}\, x\, \text{Sec}^2 y - \text{Sin}\,(x + y)$

$\partial N/\partial x = \text{Cos}\, x\, \text{Sec}^2 y - \text{Sin}\,(\boldsymbol{x} + y)$

Therefore the equation is exact

Integrating $\text{Sin}\, x\, \text{Tan}\, y + \text{Sin}\,(x + y) = C$

Example (iii)

$y\, dx - x\, dy + 3x^2 y^2 e^{(x)3}\, dx = 0$

This is not exact, but divide by y^2

$(1/y)dx - (x/y^2)\, dy + 3x^2 e^{(x)3} dx = 0$

$M = (1/y) + 3x^2\, e^{(x)3}$ and $N = -\,(x/y^2)$

$\partial M/\partial y = -(1/y^2)$ and $\partial N/\partial x = -(1/y^2)$

The equation is exact

Integrating $x/y + \int 3x^2\, e^{(x)3}\, dx + C = 0$
put $u = x^3$
$du = 3x^2\, dx$
$\int 3x^2\, e^{(x)3}\, dx = \int 3x^2\, e^{u}\, dx = \int e^{u}\, du = e^{u} = e^{(x)3}$
$x/y + e^{(x)3} + C = 0$

Separation of Variables

$M\, N\, dx = P\, Q\, dy$ (316)

Where M and P are functions of x only
And N and Q are functions of y only
The Solution is $\int (Q / N)\, dy = \int (M / P)\, dx + C$

Example (i)
$dy/dx = 2xy$
Separating the Variables $(1/y)\,dy = 2x\,dx$
Integrating $\ln(y) = x^2 + C$

Example (ii)
$\mathrm{Tan}\,x\ dy = \mathrm{Cot}\,y\ dx$
Separating the Variables $\mathrm{Tan}\,y\ dy = \mathrm{Cot}\,x\ dx$
Integrating $-\ln \mathrm{Cos}\,y = \ln \mathrm{Sin}\,x + C$
putting $C = -\ln a$ $\mathrm{Sin}\,x\,\mathrm{Cos}\,y = a$

Homogeneous Equations (First Order)

$dy/dx = f(y/x)$ (317)
Put $y = vx$ then $dy/dx = v + x\,dv/dx$
Therefore $v + x\,dv/dx = f(v)$
Separating the Variables and Integrating
$\int [1/\{f(v) - v\}]\,dv = \int (1/x)\,dx + C = \ln(x) + C$ (318)

Example (i)
$(x + y)dy + (x - y)dx = 0$
$dy/dx = (y - x)/(y + x)$
Put $y = vx$ therefore $v + x\,dv/dx = dy/dx = (v - 1)/(v + 1)$
$x\,dv/dx = (v - 1 - v^2 - v)/(v + 1) = -(v^2 + 1)/(v + 1)$
$-v\,dv/(v^2 + 1) - dv/(v^2 + 1) = dx/x$
Integrating
$-(1/2)\ln(v^2 + 1) - \mathrm{Arc\,Tan}\,v = \ln x + C$
$2\ln x + \ln(v^2 + 1) + 2\,\mathrm{Arc\,Tan}\,v + 2C = 0$

Putting $a = 2C$ and substituting for v
$\ln(y^2 + x^2) + 2\,\mathrm{Arc\,Tan}\,(y/x) + a = 0$

Example (ii)
$dy/dx = (y - x + 1)/(y + x + 5)$
This is not homogeneous but substitute $y = Y + a$ and $x = X + b$
$dY/dX = (Y + a - X - b + 1)/(Y + a + X + b + 5)$
Put $a - b + 1 = 0$ and $a + b + 5 = 0$
therefore $a = -3$ and $b = -2$
$dY/dX = (Y - X)/(Y + X)$ which is the same as Example (i)
$\ln(Y^2 + X^2) + 2\,\mathrm{Arc\,Tan}\,(Y/X) + a = 0$
$\ln\{(y + 3)^2 + (x + 2)^2\} + 2\,\mathrm{Arc\,Tan}\,\{(y + 3)/(x + 2)\} + a = 0$

Linear Equations, (first order)

$dy/dx + P\,y = Q$ where P and Q are functions of x (319)

Multiply by an integrating factor R

$R\,dy + P\,R\,y\,dx = R\,Q\,dx$

The LHS is the derivative of a product and the first term shows that this product is $R\,y$

Therefore the second term shows that $y\,dR = P\,R\,y\,dx$

Therefore $dR / R = P\,dx$

Integrating $\ln(R) = \int P\,dx$ hence $R = e^{\int P\,dx}$ (320)

Thus equations of the form (319) can always be solved by multiplying by the integrating factor $e^{\int P\,dx}$

Example (i)

$dy/dx + y \operatorname{Cot} x = \operatorname{Cosec} x$

$\ln(R) = \int \operatorname{Cot} x\,dx = \ln(\operatorname{Sin} x)$ Therefore $R = \operatorname{Sin} x$

$\operatorname{Sin} x\,dy + y \operatorname{Cos} x\,dx = dx$

Integrating $y \operatorname{Sin} x = x + C$

Example (ii)

$x \ln(x)\,dy/dx + y = 2\ln(x)$

Therefore $dy/dx + y/[x\ln(x)] = 2/x$

$\ln(R) = \int [1/\{x \ln(x)\}]\,dx$

$= \int \{1/\ln(x)\}\,d\{\ln(x)\} = \ln\{\ln(x)\}$

Therefore $R = \ln(x)$

$\{\ln(x)\}\,dy + (y/x)\,dx = (2/x)\ln(x)\,dx$

Integrating $y \ln(x) = \int (2/x)\ln(x)\,dx = [\ln(x)]^2 + C$

$y = \ln(x) + C/\ln(x)$

Linear Equations, Constant Co-efficients

$a_n\,d^ny/dx^n + a_{n-1}\,d^{n-1}y/dx^{n-1} + \ldots\ldots + a_1\,dy/dx + a_0\,y = f(x)$ (321)

written in short $F(D)\,y = f(x)$ (322)

Let $y = y_1$ be any one solution with no arbitrary constants

y_1 is called a Particular Integral (P.I.) (323)

Let $y = y_2$ be the full solution of the equation $F(D)\,y = 0$

y_2 contains the required number of arbitrary constants

and is called the Complimentary Function (C.F.)

Therefore $y = y_1 + y_2$ is a solution for $F(D)\,y = f(x)$ (324)

This solution contains the required number of arbitrary constants and is therefore the full solution.

The **Complimentary Function (C.F.)** for $F(D)\,y = 0$ has the form

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + C_3 e^{\alpha_3 x} + \ldots + C_n e^{\alpha_n x} \tag{325}$$

where $C_1, C_2 \ldots\ldots C_n$ are the arbitrary constants

$D\,C\,e^{\alpha x} = \alpha C\,e^{\alpha x}$

$D^2\,C\,e^{\alpha x} = \alpha^2 C\,e^{\alpha x}$

$D^n\,C\,e^{\alpha x} = \alpha^n C\,e^{\alpha x}$

Therefore and from (241)

$F(D)\,C\,e^{\alpha x} = e^{\alpha x}\,F(\alpha)\,C$

Therefore $F(D)\,y = 0$ when $F(\alpha) = 0$

Thus $\alpha 1 \alpha$,2 α ...n are the solutions to $F(\alpha) = 0$

$\alpha 1, \alpha 2$ ..... αn are found by substituting the value for y in $F(D)\,y = 0$

Special Cases

(i) $\alpha 1, \alpha 2 \ldots \alpha n$ all Real and Different.

The solution is as equation (325)

(ii) Conjugate Pair in the solution for α

eg $\alpha 1 = p + i q$ and $\alpha 2 = p - i q$

Therefore $y = C_1 e^{px} e^{iqx} + C_2 e^{px} e^{-iqx}$

From (185)

$y = e^{px} \{ C_1 \text{Cos } qx + i C_1 \text{Sin } qx + C_2 \text{Cos } qx - i C_2 \text{Sin } qx \}$

Put $C_1 + C_2 = A$ and $i (C_1 - C_2) = B$

$y = e^{px} (A \text{ Cos } qx + B \text{ Sin } qx)$ (326)

(iii) r equal roots $\alpha 1 = \alpha 2 = \alpha 3 ...= \alpha n$

r − 1 arbitrary constants are lost, so there must be other solutions

Try $y = V e^{\alpha 1x}$

Also $(D - \alpha 1)^r$ is a factor of F(D) and we are concerned with the solutions to;

$(D - \alpha 1)^r y = 0$

Put $y = V e^{\alpha 1 x}$

$(D - \alpha 1)^r y = (D - \alpha 1)^r \{ V e^{\alpha 1 x} \}$

From (241)

$(D - \alpha 1)^r y = e^{\alpha 1 x} (D - \alpha 1 + \alpha 1)^r \{ V \}$

$= e^{\alpha 1 x} (D)^r \{ V \}$

$= 0$ if $V = C1 + C2\,x + C3\,x^2 + ... + Cr\,x^r$

Thus the Complimentary Function for $F(D)\, y = f(x)$

with r equal roots to the equation $F(\alpha) = 0$ is of the form

C.F. is $y = C1\, e^{\alpha 1 x} + C2\, x\, e^{\alpha 2 x} + C3\, x^2\, e^{\alpha 3 x} + ... + Cr\, x^{r-1}\, e^{\alpha r x}$ (327)

The **Particular Integral (P.I.)**

(i) $f(x) = k$

From (321) the differential equation is;

$a_n\, d^ny / dx^n + a_{n-1}\, d^{n-1}y / dx^{n-1} + ... + a_1\, dy / dx + a_0\, y = f(x) = k$

Clearly $a_0 y = k$ is a solution as all higher differentials are zero

Thus a P.I. is $y = k / a_0$ (328)

(ii a) $f(x) = e^{kx}$

$F(D)\, y = e^{kx}$

Therefore $y = \{ 1 / F(D) \} e^{kx}$

From (243) $y = e^{kx} / \{ F(k) \}$ (329)

(ii b) If F(k) = 0 then the method fails Put

$F(D) = (D - k)\, \varphi(D)$

$(D - k)\, \varphi(D)\, y = e^{kx}$

$y = [1 / (D - k)] [1 / \{\varphi(D) \}]\ e^{kx}$

From (243)

$y = [1 / (D - k)] [e^{kx} / \varphi k)]$

$y = [e^{kx} / \varphi(k)] [1 / (D - k + k)] [1]\, y =$

$[e^{kx} / \varphi(k)]\, x$ (330)

(ii c) If $F(D) = (D - k)^r \varphi(D)$
Follow the same method as (ii b) to get;
$y = [e^{kx} / \varphi(k)] [1 / (D)^r] [1]$
$y = [e^{kx} / \varphi(k)] [x^r / r!]$ (331)

(iii a) $f(x) = \text{Cos } mx$ or $\text{Sin } mx$
$F(D) y = k \text{ Cos } mx$
Substitute $y = a \text{ Sin } mx + b \text{ Cos } mx$ (332)
Equate coefficients to evaluate a and b

(iii b) If F(D) has a factor $(D^2 + m^2)$ the method fails
Consider the equation;
$F(D) y = k e^{imx}$
$= k \text{ Cos } mx + i k \text{ Sin } mx$
Real part of $F(D) y = k e^{imx}$ gives P.I $F(D) y = k \text{ Cos } mx$ (333)
Complex part of $F(D) y = k e^{imx}$ gives P.I $F(D) y = k \text{ Sin } mx$ (334)

(iv) $f(x) = x^s$ where s is a positive integer
$F(D) y = x^s$
Therefore;
$y = [1 / \{F(D)\}] x^s$
Expand $\{F(D)\}^{-1}$ as a power series in D (335)
Powers higher than D^s give zero

(v) $f(x) = e^{kx} \text{ Cos } mx$ or $e^{kx} \text{ Sin } Mx$
Consider $F(D) y = e^{(k + im) x}$
Find the P.I. Let it be $y = y_1 + i y_2$
y_1 is the P.I. for $F(D) y = e^{kx} \text{ Cos } mx$
y_2 is the P.I. for $F(D) y = e^{kx} \text{ Sin } mx$ (336)

(vi) $f(x) = x^s e^{kx}$
$F(D) y = x^s e^{kx}$
For P.I. $y = [1 / F(D)] [e^{kx} x^s] = e^{kx} [1 / F(D + k)] x^s$ (337)
and expand $[1 / F(D + k)]$ as a series in D as for (335)

(vii) $f(x) = x^s e^{kx} \text{ Cos } mx$ or $f(x) = x^s e^{kx} \text{ Sin } mx$
Consider $F(D) y = x^s e^{(k + im)x}$
Proceed as for (337) and separate the Real and Complex parts

Examples on Linear Equations with constant coefficients

Example 1

$d^2y/dx^2 - 6\, dy/dx + 13\, y = 0$

Put $y = e^{ax}$ $\quad a^2 e^{ax} - 6a\, e^{ax} + 13\, e^{ax} = 0$

$a^2 - 6a + 13 = 0$ $\quad$ therefore $a = 3 + 2i$ or $a = 3 - 2i$

Therefore

$$\begin{aligned} y &= A\, e^{(3+2i)x} + B\, e^{(3-2i)x} \\ &= e^{3x}\{ A\, e^{2ix} + B\, e^{-2ix} \} \\ &= e^{3x}\{C \operatorname{Cos} 2x + D \operatorname{Sin} 2x\} \end{aligned}$$

Example 2

$$(D^4 + 2D^2 + 1)\,y = 0$$
$$(D^2 + 1)^2\,y = 0$$

Put $y = e^{ax}$

Therefore $(a^2 + 1)^2 = 0$

$a^2 = -1, -1$

$a = +i, -i, +i, -i$

Therefore $y = C_1 e^{ix} + E_1 x e^{ix} + C_2 e^{-ix} + E_2 x e^{-ix}$

$= A_1 \text{Cos}\,x + B_1 \text{Sin}\,x + x\{A_2 \text{Cos}\,x + B_2 \text{Sin}\,x\}$

Example 3

$$(D^3 + D)\,y = 3$$

C.F. $(D^3 + D)\,y = 0$

Put $y = e^{ax}$ *therefore* $a^3 + a = 0$ and $a = 0, +i, -i$

$y = C + A \text{Cos}\,x + B \text{Sin}\,x$

P.I. Try $D\,y = 3$ therefore $y = 3x$ and $D^3 y = 0$

Therefore $y = 3x$ is a P.I.

Complete Solution is C.F. + P.I.

$y = C + A \text{Cos}\,x + B \text{Sin}\,x + 3x$

Example 4

$(D^4 - 3D^3 + 3D^2 - D)\,y = 2$

C.F. $a(a-1)^3 = 0$ therefore $a = 0, 1, 1, 1$

$y = A + B e^x + C x e^x + D x^2 e^x$

P.I. Try $-D\,y = 2$ therefore $y = -2x$

$D^2 y = 0$, $D^3 y = 0$ and $D^4 y = 0$

Complete Solution is C.F. + P.I.

$y = A + B e^x + C x e^x + D x^2 e^x - 2x$

Example 5

$(D^2 + D - 6)\,y = e^x$

$(D + 3)(D - 2)\,y = e^x$

C.F. $(a + 3)(a - 2) = 0$ therefore $a = 2, -3$

$y = A e^{2x} + B e^{-3x}$

P.I. $y = [1/\{(D + 3)(D - 2)\}][e^x]$

$= [1/\{(1 + 3)(1 - 2)\}]\,e^x = -(1/4)\,e^x$

Complete Solution is

$y = A e^{2x} + B e^{-3x} - (1/4)\,e^x$

Example 6

$(D + 3)(D - 2)\,y = e^{2x}$

C.F. $y = A e^{2x} + B e^{-3x}$

P.I. $(D - 2)\,y = [1/(D + 3)][e^{2x}] = [1/(2 + 3)]\,e^{2x} = (1/5)\,e^{2x}$

$y = (1/5)[1/(D - 2)][e^{2x}] = (1/5)\,e^{2x}[1/(D + 2 - 2)][1]$

$= (1/5)\,e^{2x}[1/D][1] = (1/5)\,x\,e^{2x}$

Complete Solution is

$y = A e^{2x} + B e^{-3x} + (1/5)\,x\,e^{2x}$

Example 7

$(D - 3)^2 y = e^{3x}$

C.F. $y = A e^{3x} + B x e^{3x}$

P.I. $y = [1/(D - 3)^2] e^{3x} = e^{3x} [1/(D + 3 - 3)^2] [1]$

$= e^{3x} [1/D^2] [1] = e^{3x} x^2/2$

Complete Solution is

$y = A e^{3x} + B x e^{3x} + (1/2) e^{3x} x^2$

Example 8

$(D^2 + 6 D + 25) y = 2 \text{ Cos } 3x$

C.F. Put $y = e^{ax}$ in $F(D) = 0$ therefore $a = -3 + 4i, -3 - 4i$

$y = A e^{-3x} \text{ Cos } 4x + B e^{-3x} \text{ Sin } 4x$

P.I. Try $y = c \text{ Sin } 3x + d \text{ Cos } 3x$

Substituting;

$-9c \text{ Sin}3x - 9d \text{ Cos}3x + 18c \text{ Cos}3x - 18d \text{ Sin}3x + 25c \text{ Sin}3x + 25d \text{ Cos}3x = 2 \text{ Cos}3x$

Equating co-efficients of $\text{Cos}3x$ and $\text{Sin}3x$

$18c + 16d = 2$ and $16c - 18d = 0$

therefore $c = 9/145$ and $d = 8/145$

P.I. $y = (1/145) (9 \text{ Sin } 3x + 8 \text{ Cos } 3x)$

Complete Solution is;

$y = A e^{-3x} \text{ Cos } 4x + B e^{-3x} \text{ Sin } 4x + (1/145) (9 \text{ Sin } 3x + 8 \text{ Cos } 3x)$

Example 9

$(D^2 + 1) y = 4 \text{ Cos } x$

C.F. is $y = A \text{ Cos } x + B \text{ Sin } x$

P.I. Consider the Real part of $(D^2 + 1) y = 4 e^{ix}$

P.I. is the Real part of y

$y = 4/(D^2 + 1) e^{ix} = 4/[(D - i)(D + i)] e^{ix} = 4/(D - i) (1/2i) e^{ix}$

$y = (2/i) /(D - i) [e^{ix} .1] = (2/i) e^{ix} /[(D + i) - i] [1]$

$= -2 i e^{ix} [1/D] [1] = -2 i x e^{ix} = -2 i x (\text{Cos } x + i \text{ Sin } x)$

Real part is $y = 2 x \text{ Sin } x$

Complete Solution is

$y = A \text{ Cos } x + B \text{ Sin } x + 2 x \text{ Sin } x$

Example 10

$(D^2 + 6D + 13) y = x^2$

P.I. $y = 1/(13 + 6D + D^2) [x^2] = (1/13) [1 + (6D + D^2)/13]^{-1} [x^2]$

$= (1/13) [1 - \{(6D + D^2)/13\} + \{(6D + D^2)/13\}^2 - ....] [x^2]$

$= (1/13) [1 - 6D/13 - D^2/13 + 36D^2/169 + \text{higher orders of D}] [x^2]$

$= (1/13) (x^2 - 12 x/13 - 2/13 + 72/169)$

$= (1/13) (x^2 - 12 x/13 + 46/169)$

Integrating Factor

Linear Equations such as $d^2x/dt^2 = -A\,x$ can be integrated at once by use of an Integrating Factor.

Multiply Both sides by Integrating Factor (2 dx/dt)

$$2\,dx/dt\;d^2x/dt^2 = -2\,A\,x\,dx/dt$$

Integrating $\quad (dx/dt)^2 = -A\,x^2 + C$

$$dx/dt = \sqrt{(C - Ax^2)}$$

Separate the variables (ie put x terms on left, t terms on right)

$$\int [1 / [\sqrt{(C - Ax^2)}]\,dx = \int dt$$

This leads to a solution in the form $x = a \sin \omega t + b \cos \omega t$

Homogeneous Linear Differential Equations of order n

Standard Form;

$a_n x^n d^ny/dx^n + ... + a_2 x^2 d^2y/dx^2 + a_1 x\,dy/dx + a_0 y = f(x)$ (338)

where $a_0, a_1, a_2 .... a_n$ are constants

To solve, put $x = e^t$ and $D = d/dt$ therefore $dx/dt = e^t = x$

Therefore $x\,dy/dx = dy/dx\,.\,dx/dt = dy/dt = D(y)$

If V is any function of x,

then $x\,dV/dx = dV/dx\,.\,dx/dt = dV/dt = D(V)$

Write $V_r = x^r d^ry/dx^r$

Then $V_{r+1} = x^{r+1} d^{r+1}y/dx^{r+1} = x^{r+1} d/dx(d^ry/dx^r) = x^{r+1} d/dx(V_r/x^r)$

$V_{r+1} = x^{r+1} [x^r dV_r/dx - V_r\, r /x^{r-1}]/x^{2r} = x\,dV_r/dx - r V_r = (D - r) V_r$ (339)

But $V_1 = x\,dy/dx = D(y)$

Put r = 1 $V_2 = (D - 1) V_1 = (D - 1) D(y)$

Put r = 2 $V_3 = (D - 2) V_2 = (D - 2) (D - 1) D(y)$ (340)

Thus equation (338) can be reduced to the form

$F(D) = f(x)$

This a linear equation with constant coefficients

Example (i)

$x\,d^2y/dx^2 - 2\,dy/dx + 2\,y/x = 4\,x^2$

$x^2 d^2y/dx^2 - 2\,x\,dy/dx + 2\,y = 4\,x^3$

Put $x = e^t$ and $D = d/dt$

Thus $D(D - 1)\,y - 2\,D\,y + 2\,y = 4\,e^{3t}$

$(D^2 - 3\,D + 2)\,y = 4\,e^{3t}$

C.F. Put $y = e^{at}$ in $(D^2 - 3\,D + 2)y = 0$ Therefore $a = 1$ or 2

C.F. is $y = A\,e^t + B\,e^{2t}$

P.I. $y = \{ 4 / (D^2 - 3\,D + 2) \} \{ e^{2t} \}$

$= 4\,e^{3t} \{ 1 / (9 - 9 + 2) \} = 2\,e^{3t}$

Complete Solution is $y = A\,e^t + B\,e^{2t} + 2\,e^{3t}$ where $e^t = x$

Complete Solution is $y = A\,x + B\,x^2 + 2\,x^3$

Example (ii)

$x^2 d^2y/dx^2 - 3\,x\,dy/dx + 4\,y = 0$

Put $x = e^t$ and $D = d/dt$

$D\,(D - 1)\,y - 3\,D\,y + 4\,y = 0$ Therefore $(D - 2)^2\,y = 0$

Put $y = e^{at}$ $a = 2$ or 2

$y = A\,e^{2t} + B\,t\,e^{2t}$

But $t = \log x$ therefore $y = A\,x^2 + B\,x^2 \log x$

Example (iii)

$x^2\, d^2y/dx^2 \;+\; x\, dy/dx \;+\; y \;=\; 2 \log x$

Put $x = e^t$ and $D = d/dt$ Thus $D\,(D - 1)\, y + D\, y + y = 2\,t$

$(D^2 + 1)\, y = 2\,t$

C.F. Put $e^{a\,t}$ in $(D^2 - 1)\, y = 0$ Thus $a = i$ or $-\,i$

Thus C.F. $y = A \operatorname{Cos} t + B \operatorname{Sin} t$

P.I. $y = \{\, 1/(1 + D^2)\,\}\,\{2\,t\} = 2\,\{\, 1 - D^2 + D^4 - \ldots\,\}\,\{\, t\,\} = 2\,t$

But $t = \log x$ therefore;

Complete Solution is $y = A \operatorname{Cos}(\log x) + B \operatorname{Sin}(\log x) + 2 \log x$

Second order differential equations reduceable to first order

a) Not containing y explicitly (341)

Equation contains d^2y/dx^2 , dy/dx , x but not y

Write $dy/dx = p$, $d^2y/dx^2 = dp/dx$ and the equation becomes a differential equation of first order

b) Not containing x explicitly (342)

Equation contains d^2y/dx^2 , dy/dx , y but not x

Write $dy/dx = p$, $d^2y/dx^2 = dp/dx = (\,dp/dy\,)\,(\,dy/dx\,) = p\, dp/dx$

Example (i) $d^2y/dx^2 + dy/dx \operatorname{Cot} x = \operatorname{Cosec} x$

No term contains y therefore put $p = dy/dx$

$dp/dx + p \operatorname{Cot} x = \operatorname{Cosec} x$

Integrating factor R is given by $\ln(R) = \int \operatorname{Cot} x\, dx = \ln(\operatorname{Sin} x)$

$R = \operatorname{Sin} x$

$dp/dx \operatorname{Sin} x + p \operatorname{Cos} x = 1$

$d/dx\,(\,p \operatorname{Sin} x\,) = 1$

$p \operatorname{Sin} x = x + A$

$p = (\,x + A\,) \operatorname{Cosec} x$

But $dy/dx = (\,x + A\,) \operatorname{Cosec} x$

$y = \int (\,x + A\,) \operatorname{Cosec} x\; dx + B$

Example (ii) $y\, d^2y/dx^2 - (\,dy/dx\,)^2 = 1$

No term contains x Therefore $y\, p\, dp/dx - p^2 = 1$

$[\,p\,/\,(\,1 + p^2\,)\,]\, dp = dy\,/\,y$

$\ln(y) = (\,1/2\,)\,[\,\ln(\,1 + p^2\,)\,] + \ln(c)$

$y^2 = c^2\,(1 + p^2\,)$

$dy/dx = p = \pm \sqrt{(\,y^2/\,c^2 - 1)}$

$x = \pm \int dy\,/\,\sqrt{(\,y^2/\,c^2 - 1)} + A$

Put $y = c \operatorname{Cosh} z$ etchence $y = c \operatorname{Cosh}[\,(\,x - A)\,/\,c\,]$

Simultaneous linear Differential equations - constant coefficients

Example (i)

$$dx/dt + 5x - y = e^{t}$$
$$dy/dt - 3y + x = 4e^{-t}$$

Write $D = d/dt$

$(D + 5)x - y = e^{t}$ therefore $y = (D + 5)x - e^{t}$

$(D - 3)y + x = 4e^{-t}$

Substitute for y

$(D - 3)(D + 5)x - (D - 3)e^{t} + x = 4e^{-t}$

$(D^2 + 2D - 14)x = 4e^{-t} - 2e^{t}$

CF Put $x = e^{at}$ in $(D^2 + 2D - 14)x = 0$ therefore $a = -1□ \pm \sqrt{15}$

$a = -4.87, +2.87$

$x = A\,e^{-4.87} + B\,e^{2.87}$

PI $x = [-2/(D^2 + 2D - 14)]\,e^{t} + [4/(D^2 + 2D - 14)]\,e^{-t}$

$= e^{t}\{-2/(1 + 2 - 14)\} + e^{-t}\{4/(1 - 2 - 14)\}$

$= (2/11)\,e^{t} - (4/15)\,e^{-t}$

Complete solution for x is;

$x = A\,e^{-4.87} + B\,e^{2.87}\ (2/11)\,e^{t} - (4/15)\,e^{-t}$

But $y = (D + 5)x - e^{t}$

Complete solution for y is;

$y = 0.13\,A\,e^{-4.87t} + 7.87\,B\ s^{2.87t} + (1/11)\,e^{t} - (16/15)\,e^{-t}$

Example (ii)

$(4D + 1)\,x - (3D + 2)\,y = t$ eqtn (a)

$(D + 5)\,x - (D + 4)\,y - 0$ eqtn (b)

Eliminate x by $(D + 5)$ eqtn (a) $=$ $(4D + 1)$ eqtn (b)

$-(D + 5)(3D + 2)\,y + (4D + 1)(D + 4)\,y = (D + 5)\,t$

$(D^2 - 6)\,y = 1 + 5\,t$

CF $y = A\,e^{/6\,t} + B\,e^{-/6\,t}$

PI $y = [1/(D^2 - 6)][1 + 5t] = -(1/6)[1/(1 - D^2/6)][1 + 5t]$

$= -(1/6)[1 + D^2/6 + D^4/36 + ...][1 + 5t]$

$= -- (1/6)(1 + 5t)$

Complete solution is;

$y = A\,e^{/6\,t} + B\,e^{-/6\,t} - (1/6)(1 + 5t)$

Eliminate y from equations (a) and (b) by;

$(D + 4)$ eqtn (a) $-$ $(3D + 2)$ eqtn (b)

hence $x = C\,e^{/6\,t} + D\,e^{-/6\,t} - (1/6)(1 + 4t)$

substitution in equation (a) shows that C and D are not independent of A and B

In fact $C = \{(3/6 + 2)/(4/6 + 1)\}\,A$

and $D = \{(3/6 - 2)/(4/6 - 1)\}\,B$

22 BESSELL'S AND LEGENDRE'S EQUATIONS

Bessell's and Legendre's equations

Both these equations are special cases of second order linear equations of the form;

$$d^2y / dx^2 + P(x) \; dy/dx + Q(x)\,y = 0$$

Bessell's equation

$$x^2 \, d^2y/dx^2 + x \, dy/dx + (x^2 - n^2)\,y = 0 \qquad (343)$$

where $n = 0.\ 1,\ 2,\ 3,\ 4, \ldots$ etc

or $n = 1/2,\ 1/3,\ 1/4, \ldots$ etc

Legendre's equation

$$(1 - x^2)\, d^2y/dx^3 - 2\,x \, dy/dx + n\,(n + 1)\,y = 0 \qquad (344)$$

where $n = 0,\ 1,\ 2,\ 3,\ 4, \ldots$ etc

Singular Points

If for continuous values of x, $P(x)$ or $Q(x)$ go off to infinity (called a Singular Point), then one solution will be singular at this point.

For Bessell, $x = 0$ is a singular point

For Legendre, $x = \pm 1$ is a singular point

Solution

For solution, put $y = a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} \ldots a_r x^{c+r} + \ldots$ (345)

$$y = \sum_{r=0}^{\infty} a_r \, x^{c+r}$$

Substitute in the original equation and equate coefficients.

It can be shown that a solution of the form (345) is permissible if the singular points are Regular, ie if Limit as $x \to 0$ of $(x - a)\,P(x)$ and Limit as $x \to 0$ of $(x - a)^2\,Q(x)$ are both finite for singular point $(x = a)$

Bessell's equation

$P(x) = 1/x$ and $Q(x) = (x^2 - n^2) / x^2$

Therefore $x\,P(x) = 1$ which is finite as $x \to 0$

and $x^2\,Q(x) = x^2 - n^2$ which is finite as $x \to 0$

Therefore $x = 0$ is a Regular Singular Point

Substituting

$$y = \sum_{r=0}^{\infty} a_r\, x^{c+r}$$

where $a_0 \neq 0$

$$\sum_{r=0}^{\infty} a_r (c+r)(c+r-1)\, x^{c+r} + \sum_{r=0}^{\infty} a_r (c+r)\, x^{c+r} - n^2 a_r\, x^{c+r} + \sum_{r=0}^{\infty} a_r\, x^{c+r+2} = 0 \qquad (346)$$

equating coefficients of x^c (ie $r = 0$ in terms X^{c+r})

$a_0\, c\,(c-1) + a_0\, c - n^2 a_0 = 0$

$a_0\,(c^2 - n^2) = 0$

$c^2 = n^2$ indicial equation

$c = \pm n$ $c_1 = n$ and $c_2 = -n$ (347)

equating coefficients of x^{c+1} (ie r = 1 in terms X^{c+r})

$$a_1 (c + 1) c + a_1 (c + 1) - n^2 a_1 + 0 = 0$$
$$a_1 [(c + 1)^2 - n^2) = 0$$

if $c \neq -1/2$ then $(c + 1)^2 - n^2) \neq 0$ therefore $a_1 = 0$ (348)

equating coefficients of x^{c+2} (ie r = 2 in terms X^{c+r})

$$a_2 (c + 2) (c + 1) + a_2 (c + 2) - n^2 a_2 + a_0 = 0$$
$$a_2 [(c + 2)^2 - n^2) = -a_0 \qquad (349)$$

equating coefficients of x^{c+r}

$$[a_r (c + r) (c + r - 1) + a_r (c + r) - n^2 a_r] + a_{r-2} = 0$$
$$a_r [(c + r)^2 - n^2] = -a_{r-2} \qquad (350)$$

Thus

a_3, a_5, a_7 etc are all expressible in terms of a_1 and therefore all zero
a_2, a_4, a_6 etc are all expressible in terms of a_0 which is not zero

Consider the even coefficient a_{2s}

$$a_{2s} = -a_{2s-2} / [(c + 2s + n) (c + 2s - n)] \qquad (351)$$
$$= (-1)^2 a_{2s-4} / [(c + 2s + n)(c + 2s - n)(c + 2s - 2 + n)(c + 2s - 2 - n)]$$

But from (347) $c = \pm n$ Put $c = n$ in (351)

$$a_{2s} = -a_{2s-2} / [(2n + 2s) 2s]$$
$$= -a_{2s-2} / [2^2 (n + s) s] = (-1)^2 a_{2s-4} / [2^4 (n + s)(n + s - 1) s (s - 1)]$$
$$= (-1)^s a_0 / 2^{2s} (n + s)(n + s - 1)(n + s - 2) \ldots\ldots\ldots\ldots (n + 1) s!] \qquad (352)$$

Hence the solution for $c = n$ is;

$$y = a_0 x^n + a_2 x^{n+2} + a_4 x^{n+4} \ldots\ldots\ldots$$
$$= a_0 x^n [1 - x^2 / \{2^2 (n+1) 1\} + x^4 / \{2^4 (n+2)(n+1) 2!\} - \ldots\ldots$$
$$+ (-1)^s x^{2s} / \{2^{2s} (n+s)(n+s-1) \ldots\ldots\ldots (n+1) s!\} + \ldots\ldots \qquad (353)$$

If n is an integer, we can put;

$$a_0 = A / [2^n n!] \qquad (354)$$

Then;

$$y / A = x^n / [2^n n!] - x^{n+2} / [2^{n+2} (n+1)! 1!] + x^{n+4} / [2^{n+4} (n+2)! 2!] + \ldots$$
$$+ (-1)^s (x/2)^{n+2s} / [(n+s)! s!] + \ldots\ldots\ldots \qquad (355)$$

This Series is called $J_n(x) \equiv \sum_{s=0}^{s=\infty} \frac{(-1)^s \left[\frac{x}{2}\right]^{2s+n}}{(s+n)!\ s!}$

where $n = 0, 1, 2, 3$ etc
Hence $y = A J_n(x)$ is part of the solution of (343)
ie part where $c = +n$

Values of $J_n(x)$ when n is not an integer

n! only has a meaning if n is an integer
If n is not an integer, say n = r, then n! can be replaced by the gamma function (356)

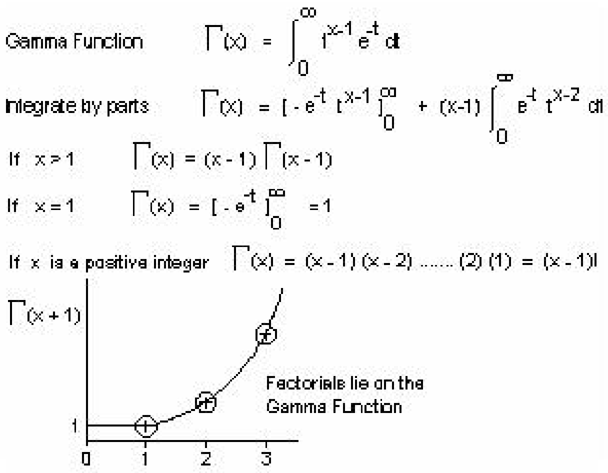


Figure 105: Gamma Function

Thus factorials containing n can be replaced by the Gamma Function and non integer values given to n.

Put $a_0 = A / [\, 2^n \, \Gamma(n + 1) \,]$ (357)

$$J_n(x) \equiv \sum_{x=0}^{x=\infty} \frac{(-1)^s \left[\frac{x}{2}\right]^{2s+n}}{\Gamma(s+n+1)\, s!} \tag{358}$$

where $n > 0$

For any given value of n and x, $J_n(x)$ can be evaluated.

Hence a family of curves can be plotted which cross and recross the X axis

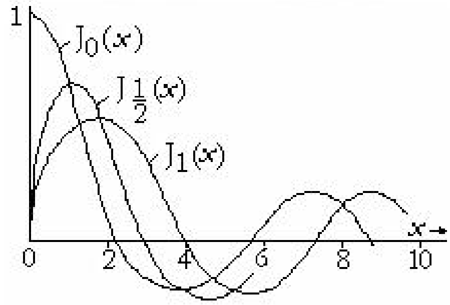


Figure 106: $J_n(x)$ functions with n reciprocal of an integer

$J_n(x) = 0$ at the following values of x

$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$
2.4	3.8	5.1	6.4
5.5	7.0	8.4	9.8
8.7	10.2	11.6	10.0
11.8	13.3	14.8	16.2
14.9	16.5	18.0	19.4
18.1	19.6	21.1	22.6

Values of $J_n(x)$ with negative values of x

$J_n(x)$ is defined in (358) using the gamma Function defined in (356)

Consider the value of the Gamma Function of negative values

Gamma Function $\Gamma(x) = \int_0^\infty t^{(x-1)} e^{-t}\, dt$

Put $x = -z$ where z is a +ive Integer

$$\Gamma(-z) = \int_0 \frac{dt}{e^t\, t^{(1+z)}} = \int_0^{t_1} \frac{dt}{e^t\, t^{(1+z)}} + \int_{t_1}^{\infty} \frac{dt}{e^t\, t^{(1+z)}}$$

But $\int_0^{t_1} \frac{dt}{e^t\, t^{(1+z)}} \to \infty$ as $t_1 \to 0$

Therefore $\Gamma(-z)$ is infinitely large

Therefore $\frac{1}{\Gamma(-z)} = 0$ (359)

Also it can be shown that $\Gamma(0)$ is infinitely large

Hence the first terms in the expansion of $J_n(x)$ are zero since;

$1 / [\Gamma(-n + s + 1)] = 0$ for $s = 0, 1, 2, \ldots\ldots\ldots, n - 1$

If n is integral and positive;

$$J_{-n}(x) = \sum_{s=n}^{s=\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+n}}{\Gamma(s-n+1)s!}$$

$$= \sum_{s=n}^{s=\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+n}}{(s-n)!\ s!}$$

Write r = s − n ie s = r + n

$$J_{-n}(x) = \sum_{r=n}^{r=\infty} \frac{(-1)^{r+n} \quad (2)}{r!(r+n)!}$$

$$= (-1)^n \sum_{r=0}^{r=\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\ \Gamma(r+n+1)!}$$

$$= (-1)^n J_n(x)$$

This relationship applies provided n is integral (360)

$J_n(x)$ and $J_{-n}(x)$ are linearly independent if n is not integral or zero.

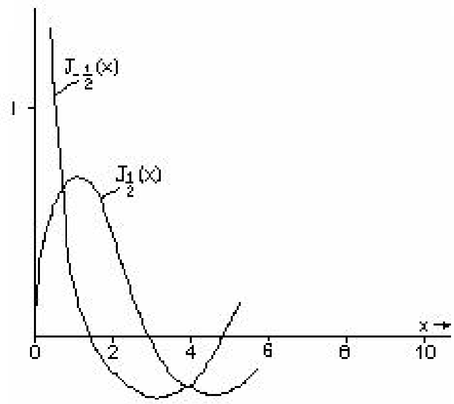


Figure 107: Jn(x) n not integral or zero

Second solution of Bessell's equation
Case (i) $0 < n < 1$ and $n \neq 1$ for example $n = 1/3$

The indicial equation had solutions $c = n$ and $c = -n$
We put $c = n$ in (351) to obtain (352) etc.

Consider now the second solution $c = -n$

Put $c = -n$ in equation (351)

$a_{2s} = -a_{2s-2} / [\,(2s-2n)\ 2s]$

$\quad = -a_{2s-2} / [\,2^2 s\,(s-n)\,] = (-1)^2 a_{2s-4} / [\,2^4 s\,(s-1)(s-n)(s-n-1)\,]$

$\quad\quad = (-1)^s a_0 / [\,2^{2s}(s-n)(s-n-1)\ldots\ldots\ldots(1-n)\ s!\,]$ (361)

But $\quad (s - n)(s - n - 1)...........(1 - n)\,\Gamma(1 - n) = \Gamma(s - n + 1)$

$\quad a_{2s} = \Gamma(1 - n)\,(-1)^s\, a_0 \,/\, [\,2^{2s}\;\Gamma(s - n + 1)\, s!\,]$

Hence the second solution is;

$$y = B\,J_{-n}(x) \tag{362}$$

where $B = a_0\,\Gamma(1 - n) \,/\, 2^n$

This does not hold for $n = 1/2$ since $(c + 1)^2 - n^2 = 0$ in (348)
Therefore a_1 is indeterminate.

Case (ii) $\;2n =$ odd integer $\;$ ie $\;n = 1/2,\ 3/2,\ 5/2$ etc
Let $\;2n = r\;$ consider $\;c = -n\;$ therefore $\;c + r = n$

Equation (322) gave;

$$a_r\,[\,(c + r)^2 - n^2\,] = -\,a_{r-2}$$

Therefore $\quad a_{r-2} = 0\;$ and $\;a_r\;$ is indeterminate

Therefore the solution with $c = -n$ contains two arbitrary constants and is the complete solution. It contains the first solution.

The Complete Solution is the same as Case (i)

Case (iii) General case where n is not an integer or zero
This includes Cases (i) and (ii)

It can be shown that the solution is the same as Case (i)

Thus for the general case where n is not an integer or zero, the Complete Solution is;

$$y = A\,J_n(x) + B\,J_{-n}(x) \qquad (363)$$

Case (iv) $n = \text{zero}$

$$J_n(x) = J_{-n}(x)$$

The Solution (335) contains only one arbitrary constant.
Hence a different solution must be found.

Case (v) $n = \text{integer}$

From (355)

$$J_{-n}(x) = (-1)^n J_n(x)$$

Hence as Case (iv), a different solution must be found.

The different solution is usually quoted as;

$$Y_n(x) = [\,\text{Cos}\,n\pi\ J_n(x) - J_{-n}(x)\,]\ /\ \text{Sin}\,n\pi \qquad (364)$$

The Complete Solution is therefore;

$$y = A\,J_n(x) + B\,Y_n(x) \qquad (365)$$

It can be seen that the solution (365) is a valid solution for cases (i), (ii) and (iii)
It can also be shown that solution (365) is also valid for Cases (iv) and (v).

For given values of n and x, $Y_n(x)$ can be evaluated. Like $J_n(x)$, it is found to be oscillatory with an infinite number of zeros and tends to zero as x tends to infinity.

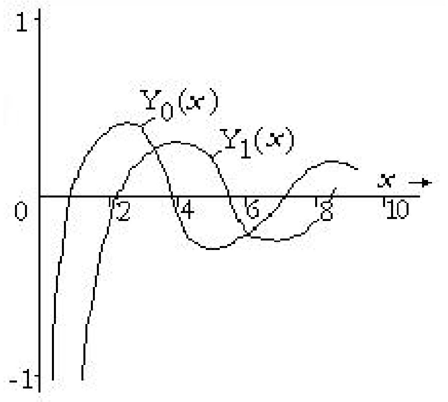


Figure 108: $Y_n(x)$

$Y_n(x) = 0$ at the following values of n and x

$Y_0(x)$	$Y_1(x)$	$Y_2(x)$	$Y_3(x)$
0.9	2.2	3.4	4.5
4.0	5.4	6.8	8.1
7.1	8.6	10.0	11.4
10.2	11.7	13.2	14.6
13.4	14.9	16.4	17.8
16.5	18.0	19.5	21.0

Summary

The solution to Bessell's equation;

$$x^2\, d^2y/dx^2 + x\, dy/dx + (x^2 - n^2)\, y = 0$$

is

$$y = A\, J_n(x) + B\, Y_n(x) \qquad (365)$$

Equations reducible to Bessell's equation

(i) $x^2\, d^2y/dx^2 + x\, dy/dx + (k^2 x^2 - n^2)\, y = 0$ (366)

Put $u = k\,x$

$dy/dx = dy/du\; du/dx = k\, dy/du$

$d^2y/dx^2 = k^2\, d^2y/du^2$

Equation becomes;

$u^2\, d^2y/du^2 + u\, dy/du + (u^2 - n^2)\, y = 0$

(ii) $d^2y/dx^2 + [(1 - 2a)/x]\, dy/dx + [(bc\, x^{c-1})^2 + (a^2 - n^2c^2)/x^2]\, y = 0$

Put $y = t\, x^a$ and $z = x^c$

Equation becomes;

$z^2\, d^2t/dz^2 + z\, dt/dz + (b^2z^2 - n^2)t = 0$ which is the same as (i) above

Example

$$d^2y/dx^2 + xy = 0$$

This is the same as (ii) above with;

$a = 1/2, \quad b = 2/3, \quad c = 3/2 \;$ and $\; n = 1/3$

Hence $y = x^{1/2} [A\, J_{1/3}(2x^{3/2}/3) + B\, Y_{1/3}(2x^{3/2}/3)]$

Further properties of Bessell Functions

(i) Consider $\quad d/dx [x^v J_v(x)] =$

$$= \frac{d}{dx}\left[\sum_{s=0}^{s=\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+v} x^v}{\Gamma(s+v+1)\, s!}\right]$$

$$= \sum_{s=0}^{s=\infty} \frac{(-1)^s \left(\frac{1}{2}\right)^{2s+v} (2s+2v)\, x^{(2s+2v-1)}}{\Gamma(s+v+1)\, s!}$$

But $\Gamma(s + v + 1) = (s + v)\,\Gamma(s + v)$

Therefore

$$d/dx\,[\,x^v J_v(x)\,] = \sum_{s=0}^{s=\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+v-1} x^v}{\Gamma(s+v)\,s!}$$

Therefore; $d/dx\,[\,x^v J_v(x)\,] = x^v J_{v-1}(x)$ (367)

(ii) Consider $d/dx\,[\,x^{-v} J_v(x)\,]$

$$= \frac{d}{dx}\left[\sum_{s=0}^{s=\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+v} x^{-v}}{\Gamma(s+v+1)\,s!}\right]$$

$$= \sum_{s=0}^{s=\infty} \frac{(-1)^s \left(\frac{1}{2}\right)^{2s+v} 2s\, x^{(2s-1)}}{\Gamma(s+v+1)\,s!}$$

$$= \sum_{s=0}^{s=\infty} \frac{-(-1)^{s-1} \left(\frac{1}{2}\right)^{2s+v-1} x^{(2s-1)}}{\Gamma(s+v+1)\,(s-1)!}$$

First term is zero, therefore limits can be from $s = 1$ to $s = \infty$
Put $r = s - 1$ therefore limits are from $r = 0$ to $r = \infty$

Therefore

$d/dx\,[x^{-v} J_v(x)\,]$

$$= \sum_{r=0}^{r=\infty} \frac{-(-1)^r \left(\frac{1}{2}\right)^{2r+v+1} x^{(2r+1)}}{\Gamma(r+v+2)\;r!}$$

$$= \sum_{r=0}^{r=\infty} x^{-v} \frac{-(-1)^r \left(\frac{x}{2}\right)^{2r+(v+1)} x^{(2r+1)}}{\Gamma(r+(v+1)+1)\;r!}$$

$d/dx\,[\,x^{-v} J_v(x)\,] = -\,x^{-v} J_{v+1}(x)$ (368)

Modified Bessell Functions

Consider the differential equation

$$x^2\, d^2y/dx^2 + x\, dy/dx - (x^2 - n^2)\,y = 0 \qquad (369)$$

Put $x = i\,t$
$dy/dx = dy/dt\; dt/dx = (-\,i)\; dy/dt$
Equation becomes;

$$t^2\, d^2y/dt^2 + t\, dy/dt + (t^2 - n^2)\,y = 0$$

This is Bessell's equation and the solution is;

$$y = A\,J_n(t) + B\;Y_n(t)$$

Substituting for t;

$$y = A\,J_n(i\,x) + B\,Y_n(i\,x)$$

The Modified Bessell Function of the first kind is defined as;

$$I_n(x) = i^{-n} J_n(i\,x)$$

$$= \sum_{r=0}^{r=\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\;\Gamma(n+r+1)} \qquad (370)$$

The Modified Bessell Function of the second kind is defined as;

$$K_n(x) = \pi/2\,[\,I_{-n}(x) - I_n(x)\,] / \text{Sin}\, n\pi \qquad (371)$$

$I_n(x)$ and $K_n(x)$ behave quite differently from $J_n(x)$ and $Y_n(x)$

They are not oscillatory.

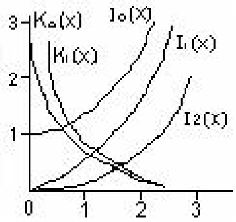


Figure 109: $I_n(x)$ and $K_n(x)$

Legendre's Equation

$$(1 - x^2)\, d^2y/dx^2 - 2x\, dy/dx + n(n + 1) = 0 \qquad (372)$$

This may be solved by substitution as was done for Bessell's equation
Alternatively the Operator $[\,x\, d/dx\,]$ can be used

These produce the solution;

$$y = a_0\, U_n(x) + a_1\, V_n(x) \qquad (373)$$

where $U_n(x) = 1 - n(n + 1)\, x^2/2! + n(n-2)(n+1)(n+3)\, x^4/4! - ....$ (374)
and $V_n(x) = x - (n-1)(n+2)\, x^3/3! + (n-1)(n-3)(n+2)(n + 4)\, x^5/5! + ...$ (375)
Both series converge if $-1 < x < 1$

If n is an even integer, $U_n(x)$ terminates
If n is an odd integer, $V_n(x)$ terminates

23 LAPLACE TRANSFORM

The Laplace transform gives an easy solution for a range of differential equations and at the same time evaluates the arbitrary constants.

The method is used for evaluating the output of a control or amplification system with various inputs all of which are zero at times before t = 0.

Laplace Transform

For any given f(t), there is a unique Laplace Transform Lf(t)

The Laplace Transform of f(t) is defined as;

$$\mathrm{L}\, f(t) = \int_0^\infty e^{-st}\, F(t)\, dt \qquad (376)$$

When Lf(t) is evaluated, it contains s but not t
Hence Lf(t) can be written as F(s)

The Laplace Transform is sometimes defined with limits $-\infty$ and $+\infty$
but in Engineering the limits are 0 and $+\infty$

Evaluate Lf(t) where
$f(t) = A\, e^{-at}$ when $t >= 0$ and $f(t) = 0$ when $t < 0$

$$L\, f(t) = \int_0^\infty A\, e^{-at}\, e^{-st}\, dt = \int_0^\infty A\, e^{-(s+a)t}\, dt = [\, \{A\, e^{-(s+a)t}\} \,/\, \{-(s+a)\}\,]_0^\infty$$

$= 0 - A/-(s + a) = A/(s + a)$

$$F(s) = A/(s + a) \tag{377}$$

Put $a = 0$
$$f(t) = A \text{ and } F(s) = A/s \tag{378}$$

Evaluate Lf(t) where $f(t) = A\, t^n\, e^{-at}$ and n is an integer

$$L\,[\, A\, t^n\, e^{-at}\,] = \int_0^\infty A\, t^n\, e^{-at}\, e^{-st}\, dt = \int_0^\infty A\, t^n\, e^{-(s+a)t}\, dt$$

Put $v = A\, t^n$ and $du = e^{-(s+a)t}\, dt$

$$L\, f(t) = [\, \{A\, t^n\, e^{-(s+a)t}\}/\{-(s+a)\}]_0^\infty - \int_0^\infty [\{Ant^{(n-1)}\, e^{-(s+a)t}\}/\{-(s+a)\,]\, dt$$

$$= [\, \{-A\, t^n\, (s+a)\, /\, (1 + t + t^2/2! + \quad)]_0^\infty + \{n/(s+a)\} \int_0^\infty A\, t^{(n-1)}\, e^{-(s+a)t}\, dt$$

$$= 0 + [n/(s + a)]\, L[A\, t^{(n-1)}\, e^{-at}]$$

$$= [n!/(s + a)^n]\, L[A\, e^{-at}]$$

$$= [n!/(s + a)^n]\, [A/(s + a)]$$

$$L\,[\, A\, t^n\, e^{-at}\,] = A\, n!/(s + a)^{n+1}$$

When $f(t) = A\, t^n\, e^{-at}$ and n is an integer Then $F(s) = A\, n!\, /\, (a + s)^{n+1}$ (379)

Laplace Transform of a differential d[f(t)]/dt

$$L\,[\,df(t)/dt] = \int_0^\infty df(t)/dt\; e^{-st}\,dt$$

$$= [f(t)\,e^{-st}]_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= 0 - F(0) + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= -\,f(0) + s\,L[f(t)]$$

Thus;

$$L[df(t)/dt] = s\,F(s) - f(0) \qquad (380)$$

where F(s) is the Laplace transform of f(t) and f(0) is the value of f(t) when t = 0

Similarly

$$L[d^2/dt^2\,f(t)] = s^2\,F(s) - sf(0) - d/dt[f(0)] \qquad (381)$$

where d/dt[f(0)] is the value of df(t)/dt when t = 0

And

$$L[d^n/dt^n\,f(t)] = s^n\,F(s) - s^{n-1}\,f(0) - \ldots - s\,d^{n-2}/dt^{n-2}[f(0)] - d^{n-1}/dt^{n-1}[f(0)] \qquad (382)$$

where $d^{n-1}/dt^{n-1}[f(0)]$ is the value of $d^{n-1}/dt^{n-1}[f(t)]$ when $t = 0$ etc

Laplace Transform of an Integral $\int [f(t)]\, dt$

Put $I = \int f(t)\, dt$

$$L[I] = \int_0^\infty I e^{-st}\, dt$$

Put $u = I$ and $v = -(1/s)\, e^{-st}$

$$L[I] = [-(1/s)\, e^{-st}]_0^\infty + \int_0^\infty [f(t)/s]\, e^{-st}\, dt$$

If $f(t) = 0$ when $t = 0$ then $I = 0$ when $t = 0$
$e^{-st} = 0$ when $t = \infty$. Therefore first term $= 0$

$$L[I] = (1/s) \int_0^\infty f(t)\, e^{-st}\, dt = (1/s)\, L[f(t)]$$

Thus $L[\int f(t)\, dt] = (1/s)\, F(s)$ where $F(s) = L[f(t)]$ (383)

Example (i) Solve the differential eqtn. $dx/dt + 3x = 6$

Take Laplace transforms

$$L[\, dx/dt + 3x] = L[6]$$

From (380), (378) and (379) with n=1 and a=0

$$s\,F(s) - x_0 + 3\,F(s) = 6/s$$
$$(s + 3)\,F(s) = 6/s + x_0$$
$$F(s) = x_0/(s + 3) + 6/[\,s\,(s + 3)]$$

Split into Partial fractions

$$F(s) = x_0/(s + 3) + 2/s - 2/(s + 3)$$

From (378) and (379)

$$x(t) = x_0\, e^{-3t} + 2 - 2\, e^{-3t}$$

The solution has evaluated the arbitrary constant in terms of the value of x when t = 0.

Example (ii)

$$(D^2 + 3D + 2)\, x = 4\, e^t$$
when $t = 0$, then $x = -1$ and $dx/dt = -1$

$$[\, s^2 F(s) + s + 1] + 3[\, s\,F(x) + 1] + 2\,F(s) = 4/(s - 1)$$
$$[s^2 + 3s + 2]\,F(s) = 4/(s - 1) - s - 4$$
$$F(s) = [\,1 / \{(s + 1)(s + 2)\}]\,\{(4 - s^2 + s - 4s + 4) / (s - 1)\}$$
$$= (-s^2 - 3s + 8) / [(s + 1)(s + 2)(s - 1)]$$
$$= A/(s + 1) + B/(s + 2) + C/(s - 1)$$

Evaluate A, B and C for the Partial Fractions;

$$F(s) = -5/(s + 1) + (10/3)/(s + 2) + (2/3)/(s - 1)$$
$$x(t) = (2/3)\, e^t - 5\, e^{-t} + (10/3)\, e^{-2t}$$

Thus the differential equation has been solved and the arbitrary constants evaluated in terms of x and dx/dt at $t = 0$.

A table of Laplace Transforms can be made up from (379) choosing values for n and a.
Re[ ] means Real Part of the complex number
Im[ ] means Complex (or Imaginary) Part of the complex number

n	a	f (t)	F (s)
n	a	$A\ t^n\ e^{-at}$	$A\ n! / (a + s)^{n+1}$
0	0	A	A / s
1	0	$A\ t$	A / s^2
	0	$A\ t^2$	$2\ A / s^3$
2	a	$A\ e^{-at}$	$A / (a + s)$
1	a	$A\ t\ e^{-at}$	$A / (a + s)^2$
0	$-\ j\omega$	$A \text{ Sin } \omega t = \text{Im} [A\ e^{j\omega t}]$	$\text{Im} [A / (s - j\omega)] = A\ \omega / (\omega^2 + s^2)$
	$-\ j\omega$	$A \text{ Cos } \omega t = \text{Re} [A\ e^{j\omega t}]$	$\text{Re} [A / (s - j\omega)] = A\ s / (\omega^2 + s^2)$
0	$-\ j\omega$	$A\ t \text{ Sin } \omega t = \text{Im} [A\ t\ e^{j\omega t}]$	$\text{Im} [A / (s - j\omega)^2] = A\ 2\ s\ \omega / (\omega^2 + s^2)^2$
1	$-\ j\omega$	$A\ t \text{ Cos } \omega t = \text{Re} [A\ t\ e^{j\omega t}]$	$\text{Re} [A / (s - j\omega)^2] = A\ (s^2 - \omega^2) / (\omega^2 + s^2)^2$
0	$a - j\omega$	$A\ e^{-at} \text{ Sin } \omega t$ $= \text{Im} [A e^{-(a- j\ \omega)t}]$	$\text{Im} [A / (a - j\omega + s)]$ $= A\ \omega / \{(\omega^2 + (s + a)^2 \}$
0	$a - j\omega$	$A\ e^{-at} \text{ Cos } \omega t$ $= \text{Re} [A e^{-(a- j\ \omega t}]$	$\text{Re} [A / (a - j\omega + s)]$ $= A(s + a) / \{\omega^2 + (s + a)^2 \}$
		$d/dt [f(t)]$	$s\ L[f(t)] - f(0)$
		$d^n/dt^n[f(t)]$	$s^n\ L[f(t)] - s^{n-1}\ f(0) - \ldots\ldots$ $-\ s\ d^{n-2}/dt^{n-2}[f(0)]$ $-\ s\ d^{n-1}/dt^{n-1}[f(0)]$
		$\int f (t)\ dt$	$(1/s)\ L[f(t)]$

24 FOURIER SERIES

Fourier Series

A Fourier Series is an infinite series that defines a cyclic function of any known shape.

Let $y = f(x)$
where $f(x)$ is the known function that is cyclic with a period of 2π

Assume that $f(x)$ can be expanded in the following series;

$$\begin{aligned} y \equiv\ & c_0 + a_1 \mathrm{Cos}\, x + a_2 \mathrm{Cos}\, 2x + a_3 \mathrm{Cos}\, 3x + \dots + a_n \mathrm{Cos}\, nx + \dots \\ & + b_1 \mathrm{Sin}\, x + b_2 \mathrm{Sin}\, 2x + b_3 \mathrm{Sin}\, 3x + \dots + b_n \mathrm{Sin}\, nx + \dots \end{aligned} \qquad (384)$$

(i) Integrate the Series with respect to x between the limits 0 and 2π

$$\begin{aligned} \int y\, dx &= 2\pi c_0 + a_1 \int \mathrm{Cos}\, x\, dx + \dots + a_n \int \mathrm{Cos}\, nx\, dx + \dots \\ &\qquad + b_1 \int \mathrm{Sin}\, x\, dx + \dots + b_n \int \mathrm{Sin}\, nx\, dx + \dots \\ &= 2\pi c_0 + a_1 [\,\mathrm{Sin}\, x\,] + \dots + a_n [\,(1/n)\, \mathrm{Sin}\, nx\,] + \dots \\ &\qquad + b_1 [\,-\mathrm{Cos}\, x\,] + \dots + b_n [\,(-1/n)\, \mathrm{Cos}\, nx\,] + \dots \end{aligned} \qquad (385)$$

With Limits 0 and 2π all the Sin terms are zero and all the Cos factors are unity so cancel each other, therefore;

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, dx \tag{386}$$

(ii) Multiply the Series (384) by Cos (r x) and integrate between 0 and 2π

$$\int_0^{2\pi} y \operatorname{Cos} r x\, dx = C_0 \int_0^{2\pi} \operatorname{Cos} r x\, dx + a_1 \int_0^{2\pi} \operatorname{Cos} x \operatorname{Cos} r x\, dx + \cdots$$

$$+ a_r \int_0^{2\pi} \operatorname{Cos}^2 r x\, dx + \cdots + a_n \int_0^{2\pi} \operatorname{Cos} r x \operatorname{Cos} n x\, dx + \cdots$$

$$+ b_1 \int_0^{2\pi} \operatorname{Cos} r x \operatorname{Sin} x\, dx + \cdots + b_r \int_0^{2\pi} \operatorname{Cos} r x \operatorname{Sin} r x\, dx + \cdots + b_n \int_0^{2\pi} \operatorname{Cos} r x \operatorname{Sin} n x\, dx + \cdots$$

But (387)

$$C_0 \int_0^{2\pi} \operatorname{Cos} r x\, dx = C_0 \left[\frac{1}{r} \operatorname{Sin} r x\right]_0^{2\pi} = [0 - 0] = 0 \tag{388}$$

And

$$a_n \int_0^{2\pi} \operatorname{Cos} r x \operatorname{Cos} n x\, dx = a_n \frac{1}{2} \int_0^{2\pi} [\operatorname{Cos}(n+r)x + \operatorname{Cos}(n-r)x]\, dx$$

$$= a_n \frac{1}{2} \left[\frac{1}{n+r} \operatorname{Sin}(n+r)x + \frac{1}{n-r} \operatorname{Sin}(n-r)x\right]_0^{2\pi} = a_n \frac{1}{2} [0 - 0 + 0 - 0] = 0 \tag{389}$$

And

$$a_r \int_0^{2\pi} \operatorname{Cos}^2 r x\, dx = a_r \frac{1}{2} \int_0^{2\pi} [\operatorname{Cos}(2r x) + 1]\, dx$$

$$= a_r \frac{1}{2} \left[\frac{1}{2r} \operatorname{Sin}(2r x) + x\right]_0^{2\pi}$$

$$= a_r \frac{1}{2} [0 - 0 + 2\pi - 0] = a_r \pi \tag{390}$$

And

$$b_n \int_0^{2\pi} \operatorname{Cos} rx \operatorname{Sin} nx\,dx = b_n \frac{1}{2} \int_0^{2\pi} [\operatorname{Sin}(n+r)x + \operatorname{Sin}(n-r)x]\,dx$$

$$= b_n \frac{1}{2}\left[\frac{-1}{n+r}\operatorname{Cos}(n+r)x - \frac{1}{n-r}\operatorname{Cos}(n-r)x\right]_0^{2\pi} = b_n \frac{1}{2}\left[\frac{-1}{n+r} - \frac{-1}{n+r} + \frac{-1}{n-r} - \frac{-1}{n-r}\right] = 0$$

(391)

And

$$b_r \int_0^{2\pi} \operatorname{Cos} rx \operatorname{Sin} rx\,dx = b_r \frac{1}{2} \int_0^{2\pi} \operatorname{Sin} 2rx\,dx$$

$$= b_r \frac{1}{2}\left[\frac{-1}{2r}\operatorname{Cos} 2rx\right]_0^{2\pi} = b_r \frac{1}{2}\left[\frac{-1}{2r} - \frac{-1}{2r}\right] = 0$$

(392)

Therefore integrating the series [(384) × Cos rx] gives (387). The terms of (387) are evaluated by (388) to (392). These show that all terms are zero except the term involving $\operatorname{Cos}^2 rx$. Hence the coefficient a_r of this term can be evaluated

$$\int_0^{2\pi} y \operatorname{Cos} rx\,dx = \pi a_r$$

$$a_r = \frac{1}{\pi} \int_0^{2\pi} y \operatorname{Cos} rx\,dx$$

(393)

The coefficient b_r can be evaluated in the same way.
Multiply the series (384) by Sin r x and integrate from 0 to 2π

$$\int_0^{2\pi} y \operatorname{Sin} rx\,dx = C_o \int_0^{2\pi} \operatorname{Sin} rx\,dx + a_1 \int_0^{2\pi} \operatorname{Cos} x \operatorname{Sin} rx\,dx + \cdots$$

$$+ a_r \int_0^{2\pi} \operatorname{Cos} rx \operatorname{Sin} rx\,dx + \cdots + a_n \int_0^{2\pi} \operatorname{Sin} rx \operatorname{Cos} nx\,dx + \cdots$$

$$+ b_1 \int_0^{2\pi} \operatorname{Sin} rx \operatorname{Sin} x\,dx + \cdots + b_r \int_0^{2\pi} \operatorname{Sin}^2 rx\,dx + \cdots + b_n \int_0^{2\pi} \operatorname{Sin} rx \operatorname{Sin} nx\,dx + \cdots$$

(394)

This is of a similar form to (387). It can be seen that all terms evaluate to zero except the term involving y and the term involving $\operatorname{Sin}^2 rx$. Hence the coefficient b_r can be evaluated.

$$\int_0^{2\pi} y \,\text{Sin}\, r x \,dx = b_r \int_0^{2\pi} \text{Sin}^2\, r x \,dx = b_r \frac{1}{2} \int_0^{2\pi} [1 - \text{Cos}\,(2r\, x)]\,dx$$

$$= b_r \frac{1}{2} \left[x - \frac{1}{2r} \text{Sin}(2r\, x) \right]_0^{2\pi} = b_r \frac{1}{2} [\,2\pi - 0 + 0 - 0] = b_r \pi$$

$$b_r = \frac{1}{\pi} \int_0^{2\pi} y \,\text{Sin}\, r x \,dx \tag{395}$$

Thus if y is any function of x, then y can be expressed as the Fourier Series

$$y = c_0 + a_1 \text{Cos}\, x + a_2 \text{Cos}\, 2x + \ldots\; a_n \text{Cos}\, nx + \ldots$$
$$+ b_1 \text{Sin}\, x + b_2 \text{Sin}\, 2x + \ldots\; b_n \text{Sin}\, nx + \ldots \tag{396}$$

where

$$C_0 = \frac{1}{2\pi} \int_0^{2\pi} y \,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} y \,\text{Cos}\, n x \,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} y \,\text{Sin}\, n x \,dx \tag{397}$$

***y* symmetrical about X-axis**

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx$$

C_0 is the mean value of y over one cycle
Therefore if y is symmetrical about the x-axis then $C_0 = 0$

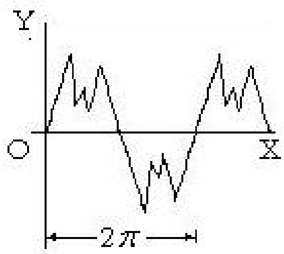


Figure 110: y symetrical about the X axis

***y* symmetrical about the Y-axis**

If y is symmetrical about the Y axis and is expressed as an ascending series of x then it contains only even powers of x.

But Sine terms contain only odd powers of x when expressed as a series. Thus the sum of the Sine terms is zero

The series is

$$y = C_0 + \sum_{n=0}^{\infty} a_n \operatorname{Cos} nx$$

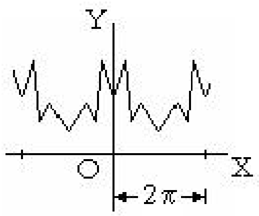


Figure 111: y symetrical about the Y axis

Curve unchanged when rotated 180 degrees about the point $x = 0$, $y = C_0$

If $(y - C_0)$ and x values are the same as $-(y - C_0)$ and $-x$ values,
ie the curve is unchanged when rotated through 180 degrees about point $x = 0$, $y = C_0$
Cosine terms are all zero

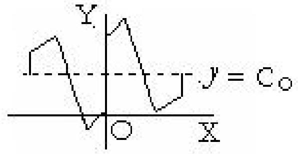


Figure 112: Curve unchanged when rotated 180 degrees

Cyclic nature of Fourier Series

Fourier Series is

$$y = Co + \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

Put $x = z + 2\pi$

$$y = Co + \sum_{n=0}^{\infty} a_n \cos n(z + 2\pi) + \sum_{n=0}^{\infty} b_n \sin n(z + 2\pi)$$

$$y = Co + \sum_{n=0}^{\infty} a_n \cos nz + \sum_{n=0}^{\infty} b_n \sin nz$$

Thus the series is cyclic with a period $x = 2\pi$

Example (i)

Find the Fourier Series to express the waveform shown here where $y = 1 + x/L$

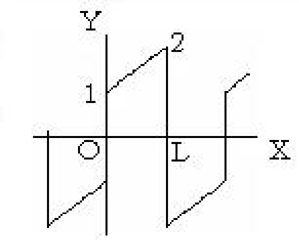


Figure 113: $y = 1 + x/L$

Put $z = x + \pi x/L$
$y = 1 + z/\pi$ while $0 < z < \pi$

The period is now 2π

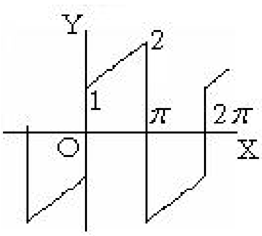


Figure 114: Modified wave form with Period 2π

By symmetry, the series contains Sine terms only
Let $y = b_1 \sin z + b_2 \sin 2z + \ldots + b_n \sin nz + \ldots$

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} y \ \text{Sin}\, nz\, dz \\
&= \frac{2}{\pi}\int_0^{\pi} (1+\frac{z}{\pi})\ \text{Sin}\, nz\, dz \\
&= \frac{2}{\pi}\Big[(1+\frac{z}{\pi})(-\frac{1}{n}\text{Cos}\, nz)\Big]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}(-\frac{1}{n}\text{Cos}\, nz)(\frac{1}{\pi})dz \\
&= \frac{2}{\pi}\Big[-\frac{2}{n}\text{Cos}\, n\pi + \frac{1}{n}\Big] + \frac{2}{n\pi^2}\Big[\frac{1}{n}\text{Sin}\, nz\Big]_0^{\pi} \\
&= \frac{2}{n\pi} - \frac{4}{n\pi}\text{Cos}\, n\pi
\end{aligned}
$$

Thus when n is odd, $b_n = 6/n\pi$ and when n is even $b_n = -2/n\pi$

Fourier Series
$y = \Sigma b_n \text{Sin}\, nz = \Sigma b_n \text{Sin}\,(n\pi x/L)$

$y = 2/\pi[3 \text{Sin}(\pi x/L) - 1/2 \text{Sin}(2\pi x/L) + 3/3 \text{Sin}(3\pi x/L) - 1/4 \text{Sin}(4\pi x/L) + ..$

This series is not true when $x = 0$ or $x = L$. Generally at points where the periodic function is discontinuous, the Fourier Series gives the mean value of the periodic function.

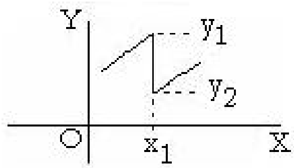


Figure 115: Discontinuous point

When $x = x_1$ the Fourier Series gives
$y = ½ (y_1 + y_2)$

Example (ii)

Find the series to express the periodic function

$y = 1 + x/\pi$ when $0 < x < \pi$
and $y = -x/\pi$ when $\pi < x < 2\pi$

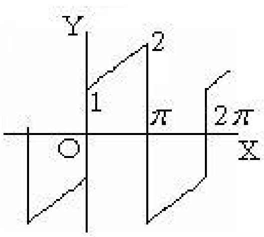


Figure 116: Periodic Function

By inspection, $C_0 = 0$

Let $y = \Sigma\, a_n \text{Cos } nx + \Sigma\, b_n \text{Sin } nx$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \left(1 + \frac{x}{\pi}\right) \text{Cos } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \left(-\frac{x}{\pi}\right) \text{Cos } nx\, dx$$

and

$$b_n = \frac{1}{\pi}\int_0^{\pi} \left(1 + \frac{x}{\pi}\right) \text{Sin } nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \left(-\frac{x}{\pi}\right) \text{Sin } nx\, dx$$

Hence;

$a_n = 2/(n^2\pi^2)\,(\text{Cos } n\pi - 1)$ and $b_n = 3/(n\pi)\,(1 - \text{Cos } n\pi)$

$$y = -4/\pi^2 \,[\,\text{Cos } x + 1/3^2 \text{ Cos } 3x + 1/5^2 \text{ Cos } 5x + \ldots\,]$$
$$+ 6/\pi\,[\,\text{Sin } x + 1/3 \text{ Sin } 3x + 1/5 \text{ Sin } 5x + \ldots\,]$$

PART 1: APPLIED MATHEMATICS

25 MECHANICS' ELEMENTARY PRINCIPLES

Statics and Dynamics

Statics is the study of bodies at rest, Dynamics is the study of bodies in motion.

Distance, Velocity and Acceleration

An object in motion moves a Distance (eg in metres) from its starting point. The Rate of Change of Distance is its speed, ie its Velocity (eg in metres/sec or m/s). The Rate of Change in Velocity is its Acceleration (eg in metres/sec/sec or m/s^2).

Thus $v = dx/dt$ $\qquad a = dv/dt = d^2x/dt^2$
Also $a = dv/dt = \text{Limit}\ [\delta v/\delta t] = \text{Limit}\ [(\delta v/\delta x)\ (\delta x/\delta t)] = \text{Limit}\ [\ v\ \delta v/\delta x]$
Thus an alternative value for acceleration is $a = v\ dv/dx$

Equations of Motion for a body moving in a straight line with Constant Acceleration

Initial velocity = u
Final velocity = v
Distance travelled = s
Time taken = t
Acceleration = a

By definition of acceleration; $dv/dt = a$
Therefore $\int dv$ from u to v = $a \int dt$ from 0 to t
$v - u = a\,t$
$v = u + a\,t$ (A1)

Distance travelled = average velocity times time
initial velocity = u
Final velocity = $v = u + a\,t$
Average velocity = $\frac{1}{2}(u + v) = u + \frac{1}{2} a\,t$

Therefore;
$s = \frac{1}{2}(u + v)\,t$ (A2)

And;
$s = u\,t + \frac{1}{2} a\,t^2$ (A3)

Multiply by (2 a)
$2\,a\,s = 2\,u\,a\,t + (a\,t)^2$
But from (M 1), $a\,t = v - u$
Therefore
$2\,a\,s = 2\,u\,(v - u) + (v - u)^2 = 2\,u\,v - 2\,u^2 + v^2 - 2\,u\,v + u^2 = v^2 - u^2$
Therefore
$v^2 = u^2 + 2\,a\,s$ (A4)

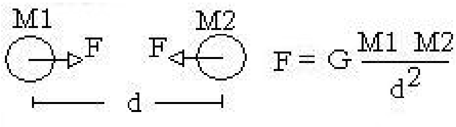
$s = u\,t + \frac{1}{2} a\,t^2$ and $v = u + a\,t$
Eliminate $u = v - a\,t$
$s = v\,t - a\,t^2 + \frac{1}{2} a\,t^2 = v\,t - \frac{1}{2} a\,t^2$
$s = v\,t - \frac{1}{2} a\,t^2$ (A5)

Equations (A1) to (A5) give the relation between Distance, Time, Speed and Acceleration.

These equations only apply for an object moving in a straight line with constant acceleration.

Gravitational Force

By studying the motion of planets, Sir Isaac Newton deduced that all bodies attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.



(A6)

Figure A1: Gravitational Constant
where G is a universal Gravitational Constant

Thus all bodies on the surface of the earth are attracted to the centre of the earth with a force proportional to their mass. Galileo dropped objects from the tower of Pisa and showed that they accelerated towards the ground with a constant acceleration. Experiments have showed that in a vacuum all bodies accelerate at the same constant rate. In a vacuum, a feather and a lump of lead will fall side by side. This acceleration, called "g", has been measured and is approximately 9.81 metres per second per second. [In fact there are very slight variations at different places of the world depending on the density of rocks near the surface].

Hence Newton deduced his 1st and 2nd Laws of Motion

Newton's 1st Law of Motion

A body continues at rest or in uniform motion in a straight line unless acted upon by a Force.

Newton's 2nd Law of Motion

A body acted upon by a steady Force has constant Acceleration.
This has been amplified to;
The Rate of Change of Momentum of a body is proportional to the Impressed Force, where Momentum is Mass times Velocity

Therefore Newton's Second Law can be written;

$$F = d/dt[\, M\, v\,] \qquad \text{(A7)}$$

Integrating, if a constant Force F is applied for a time t then

$$F\,t = \text{Change in Momentum}$$
$$= \text{Mass times change in velocity}$$
$$F\,t = M\,(v_2 - v_1) \qquad \text{(A 8)}$$

M is constant and Rate of Change of Velocity is Acceleration,
therefore $F = d/dt[\, M\, v\,]$ can be written;

$$F = M\,a \qquad \text{(A9)}$$

And the Force acting on a body due to Gravity is given by

$$F = M\,g \qquad \text{(A10)}$$

The MKS unit of Force is the Newton.
1 Newton will accelerate 1 Kg at 1 Metre/sec/sec
(A Newton is about the weight of an apple)

Action and Reaction, Newton's 3rd Law

Sir Isaac Newton deduced that for every Action there is an equal and opposite Reaction, Newton's 3rd Law.

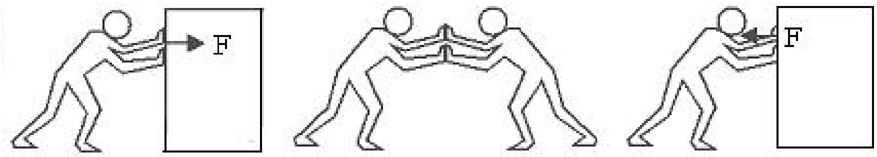


Figure A2: Newtons 3rd law

If the man pushes the box and the box is suddenly removed, he will fall over. He would need a similar man to push as hard to hold him up. Thus Newton deduced that the box pushes back with an equal and opposite force on the man.

Work

If the man moves the box, he is said to do work.
If he pushes with a steady force F for a distance X, he does work = F **x** X.
Work = Force **x** Distance
The MKS unit of Work is the Joule
1 Joule = 1 Newton **x** 1 Metre

Power

A more powerful man will move the box more quickly than a weaker man. Power is the Rate of doing Work.
Power = d/dt(Work)
The MKS unit of Power is the Watt
1 Watt = 1 Joule / sec
The Imperial unit of Power is the Horse Power, approximately the rate at which a strong horse can do work. 1 HP is approximately 746 Watts

Conservation of Energy

Energy (ie Work Done) = Force **x** Distance. It can take many forms. Lifting a weight to the top of a building gives it Energy which can be released by lowering the weight on a rope and using the rope to drive machinery.

Potential Energy.

When an object is raised above the ground, it is said to have **Potential Energy.** The energy can be used when the object is lowered back to the ground.

Kinetic Energy

When an object is moving, it is said to have **Kinetic Energy**.

When the brakes are applied on a car, the Kinetic Energy is converted into Heat. **Heat** is a form of Energy. Another form of Energy is **Sound**.

The Principle of Conservation of Energy states that Energy can be converted from one form into another but the total remains unchanged.

If an object with Mass M falls from rest a vertical distance x the Potential Energy is converted into Kinetic Energy E.

The Force on the body is M g and this is applied for a distance x
Therefore Kinetic Energy = Work Done = M g x

But the body has moved from rest with a constant Acceleration g
$v^2 = u^2 + 2as$
Initial velocity $u = 0$, acceleration $a = g$, distance fallen $s = x$ and final velocity = v
Therefore $v^2 = 2\,g\,x$

Work Done = $M\,g\,x = (½)\,M\,v^2$
Work Done has been converted into Kinetic Energy

$$\text{Kinetic Energy} = (½)\; M\,v^2 \qquad \text{(A11)}$$

Conservation of Momentum

If two objects collide, they can be damaged by the collision and Energy is used in the deformations. However during the collision, the Action on one object is equal and opposite to the Reaction on the other, (Newton's 3rd Law).

Therefore the Change in Momentum in one body is equal and opposite to the Change in Momentum in the other.

Thus the total Momentum in any direction is the same after the collision as it was before. This is the principle of the Conservation of Momentum.

Suppose a body mass M_1 and velocity v_1 collides head on with a body mass M_2 and velocity v_2 towards it. If they combine then after the collision the velocity of the combined mass is

$$V = [M_1\,v_1 - M_2\,v_2]/\,[M_1 + M_2] \qquad \text{(A12)}$$

Collisions of elastic objects

If two steel balls collide head on, they each bounce back. Little or no energy is absorbed by the collision. Newton suggested a measure of the elasticity of the objects as;

Coefficient of Restitution = e
where Relative Velocity of objects towards each other after the impact
= – e times their Relative Velocity before the impact

Thus fully elastic objects absorbing no energy have e = 1
Inelastic objects (eg a pad of butter) have e = 0 and the objects join after impact.

Let u_1 and u_2 be the velocities of objects with mass m_1 and m_2 resolved in the direction of impact before the collision and v_1 and v_2 their velocities in this direction after the collision, then;
Conservation of Momentum gives

$$m_1\,v_1 + m_2\,v_2 = m_1\,u_1 + m_2\,u_2 \qquad \text{(A13)}$$

Coefficient of Restitution gives

$$v_1 - v_2 = -e\,(u_1 - u_2) \tag{A14}$$

Hence $v_1 = [m_1 u_1 + m_2 u_2 + e\, m_2 (u_2 - u_1)] / (m_1 + m_2)$
And $v_2 = [m_1 u_1 + m_2 u_2 + e\, m_1 (u_1 - u_2)] / (m_1 + m_2)$

If two smooth spheres meet with a glancing collision, then resolve each velocity into its component parallel to the line joining their centres at impact and its component perpendicular to this line.

Example on equations of motion

A car mass 1000 kg has an engine developing 80 kW.
It can reach a speed of 100 mph on the flat
At 100 mph let the resistance be 70% wind resistance and 30% rolling resistance
Assume that wind resistance is proportional to $(\text{speed})^2$ and rolling resistance is constant

(i) Calculate the maximum speed up a 20 degree slope
100 mph = 44.7 m/sec
At 44.7 m/s the engine develops 80000/44.7 = 1790 Newton thrust
Let wind resistance be $k\ v^2$ Newton
Wind resistance $kv^2 = 44.7^2\ k = (70/100)1790 = 1250$ Newton
$k = 1250/44.7^2$ therefore $k = 0.62$
Rolling Resistance = (30/100) x 1790 = 540 Newton
Weight of the car on the slope exerts a force of Mg Sin(20) = 3350 Newton
Force to push the car up the slope at speed $v = 3350 + 540 + 0.62\ v^2$ Newton
Thrust available from the engine = 80000/v Newton
Therefore $80000/v = 3890 + 0.62\ v^2$
By trial and error, v = 19.4 m/s = 44 mph

(ii) Starting from standstill ignoring wind resistance and rolling resistance, assuming a variable gearbox and no wheel spin, find the theoretical relation between time and speed on flat ground.
Acceleration = dv/dt
M dv/dt = Thrust available from the engine = 80000/v Newton
M = 1000 Therefore dv/dt = 80/v
Multiply by v and separate the variables, put terms of v on the left and terms of t on the right
v dv = 80 dt
Integrate $(1/2)\ v^2 = 80\ t + \text{constant}$
$v^2 = 160\ t + \text{constant}$
When t = 0 then v = 0 therefore constant = 0
$v^2 = 160\ t$

(iii) Find the theoretical time to reach 60 mph from a standing start on the flat.
60 mph = 26.8 metres/second
t = 4.5 seconds

The time with wind and rolling resistance can be calculated by the computer program
v=0:t=0:dv=0.1:WHILE:v<26.8:dt=v*dv/(80-0.54*v0.00062*v^3):t=t+dt:v=v+dv:WEND:PRINT t. This gives the time with wind and rolling resistance as 5.5 seconds
This is with an infinitely variable gearbox. With a practical gearbox, the time will be longer.

(iv) Starting from standstill find the theoretical relation between speed and distance travelled on flat ground ignoring wind resistance, rolling resistance, wheel spin and assuming a perfect gear box.
Acceleration = dv/dt = dv/dx . dx/dt = v dv/dx
M v dv/dx = Thrust available from the engine = 80000/v
M = 1000
Therefore v dv/dx = 80/v
Multiply by v and separate the variables to put terms of v on the left and terms of x on the right

$[v^2 / 80]\, dv = dx$
Integrate
$(1/80) \int v^2\, dv = \int dx$
$(1/80)(1/3)\ v^3 = x + \text{constant}$
$(1/240)\, v^3 = x + \text{constant}$
$v = 0$ when $x = 0$
constant = 0
$x = (1/240)\, v^3$

(iv) Calculate the distance travelled to reach a speed of 60 mph on flat ground assuming no wind or rolling resistance.
60 mph = 26.8 m/s
$x = 80$ m

The distance with wind and rolling resistance can be calculated by the computer program
v=0:x=0:dv=0.1:WHILE:v<26.8:dx=v^2*dv/(80-0.54*v-0.00062*v^3):x=x+dx:v=v+dv:
WEND:PRINT x. This gives the distance with wind and rolling resistance as 101 metres
This is with an infinitely variable gearbox. The distance is increased by a practical gearbox..

26 ROTATIONAL MOTION

Centre of Gravity

Every object has a Centre of Gravity, called the CG.

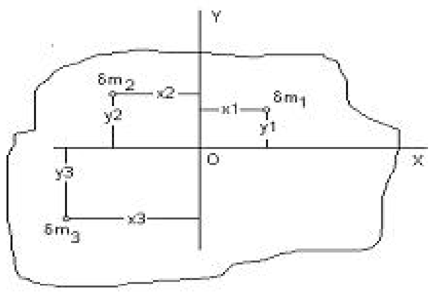


Figure A3: Centre of Gravity

The body will balance on any knife edge that passes directly below the Centre of Gravity.

The sum of the moments of each element of mass in the body about the CG = 0
In the diagram;
Choose OX and OY so that O is at the CG
The moment of δm_1 about OY is $\delta m_1 \, x_1$ and the moment of δm_1 about OX is $\delta m_1 \, y_1$

Thus for a laminar body

$$\Sigma \, [\, \delta m_1 \, x_1 \,] = 0 \quad \text{and} \quad \Sigma \, [\, \delta m_1 \, y_1 \,] = 0 \qquad \text{(A15)}$$

where x_1 and y_1 are the distances of the element δm_1 from axes through the CG
In the diagram, $x2$, $x3$ and $y3$ all have negative values

In general if O is not at the CG, then

$$\Sigma \, [\, \delta m_1 \, x_1 \,] = M \, X$$

and $$\Sigma \, [\, \delta m_1 \, y_1 \,] = M \, Y \qquad \text{A16)}$$

where X and Y are the coordinates of the CG
and M is the total mass of the body

Couple (ie Torque)

Let a body be subjected to two equal Forces F which act in opposite directions and are a distance a apart.

The Moment of the Forces about any Point P on the body is;

$$C = F(a + x) - F\,x$$
$$= F\,a \qquad \text{(A17)}$$

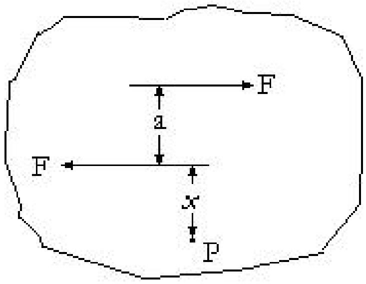


Figure A4: Couple

When the Resultant of all the Forces is zero, the Couple is the same at all points in or outside the body.

The body is said to be subjected to a **Couple** of magnitude F a

The Resultant Force in any direction due to the Couple is zero.

The Turning Force is called a Couple in Mathematics and the **Torque** in Engineering.
If half the Force is applied at twice the distance apart, the Couple is the same.

Work done by a Couple

Work Done by a Force F acting at a radius a from an axis is

$$\text{Work Done} = Fa\,\theta$$

where θ is the angle through which the Force has turned.

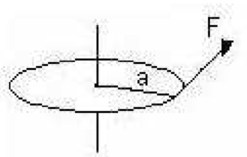


Figure A5: Work Done by a Couple

But Fa is the twisting Force, ie the Couple.

Thus replace Fa by the Couple C

$$\text{Work Done} = C\,\theta \qquad \text{(A18)}$$

Rotational Energy

A spinning flywheel certainly has kinetic energy but has no linear velocity.
Assume the flywheel has a mass M all concentrated at radius R from the centreline of the axis.
Assume the flywheel is spinning with an angular velocity of ω radians per second.

The velocity of each particle of mass δM at radius R is
$v = R\,\omega$

The Kinetic Energy of each particle is $\frac{1}{2}\,\delta M\,(R\,\omega)^2$
ω is the same for all particles
The Kinetic Energy of the flywheel is $\frac{1}{2}\,\Sigma(\delta M\,R^2)\,\omega^2$

Moment of Inertia

The quantity $\Sigma(\delta M\,R^2)$ is called Moment of Inertia and is written as capital I
The Moment of Inertia of a body is the sum of all particles of mass in the body each particle multiplied by the square of its distance from the axis.

Thus $$\text{Kinetic Energy} = \frac{1}{2}\,I\,\omega^2 \qquad \text{(A19)}$$

Radius of Gyration

The Radius of Gyration is the name given to a fictional radius such that;
Moment of Inertia = Mass times (Radius of Gyration)^2

Examples showing the method for calculating the Moment of Inertia.

(1) Moment of Inertia of a bar length 2A spinning about an axis through its centre

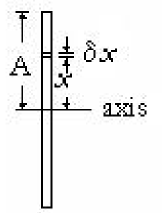


Figure A6: Moment of Inertia of a bar length 2A

$\delta I = \delta M\, x^2$
$\delta M = M\,(\delta x/2A)$
$\delta I = M\,(\delta x/2A)\, x^2$
$\delta I = M/(2A)\, x^2\, \delta x$
$I = [M/(2A)] \int x^2\, dx$ from $x = -A$ to $x = A$
$= [M/(2A)]\,[1/3]\,[\,A^3 - (-A)^3\,]$
$= M A^2/3$

(2) Moment of Inertia of a disc radius A spinning about an axis through its centre at right angles to the disc

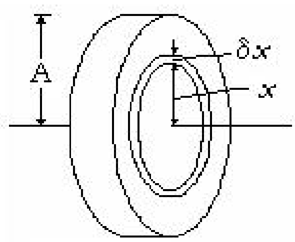


Figure A7: Moment of Inertia of a Disc

$\delta I = [(M/\pi A^2)]\; 2\,\pi\, x\, \delta x\;\; x^2$
$I = [(M/\pi A^2)] \int 2\,\pi\, x^3\, dx$ from $x = 0$ to $x = A$
$= (2M / A^2) \int x^3\, dx$ from $x = 0$ to $x = A$
$= (2M / A^2)\,]\,[\,A^4 / 4 - 0]$
$I = M A^2 / 2$

Routh's Rule

Routh's Rule states that;
Moment of Inertia
I = (Mass times sum of squares of perpendicular semi-diameters) divided by N

where $N = 3$ for rectangular laminas
$= 4$ for circular and elliptical laminas
$= 5$ for spheres and ellipsoids

Examples

1) Moment of Inertia of a Rectangular plate
about an axis AB through the centre and perpendicular to the plate.

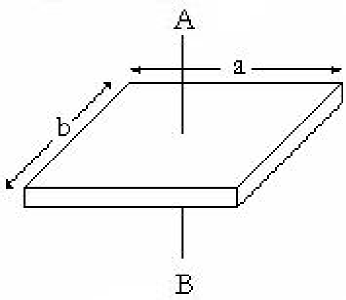


Figure A8: Moment of Inertia of a rectangular plate

$$I = M (a^2/4 + b^2/4) / 3$$
$$= M (a^2 + b^2) / 12$$

2) Moment of Inertia of a plate, length L about an axis AB along one edge.

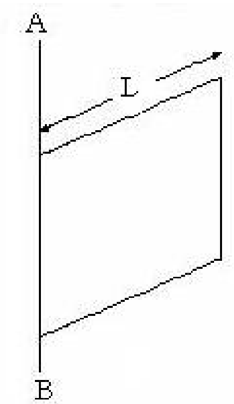


Figure A9: Moment of Inertia of a plate about one side

$$I = M (L^2) / 3$$

3) Moment of Inertia of a sphere, radius r, about an axis through the centre

$$I = M (r^2 + r^2) / 5$$
$$= 2 M r^2 / 5$$

Change of axis

Consider a laminar body in the plane OXY
δI of element δm about OX axis is $\delta I = \delta m\, y^2$
δI of element δIm about OY axis is $\delta I = \delta m\, x^2$
δI of element δIm about OZ axis is $\delta I = \delta m\, r^2$

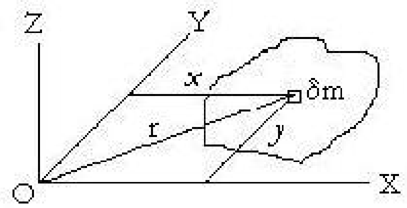


Figure A10: Change of Axis

$r^2 = x^2 + y^2$
Therefore Moment of Inertia of the element about OZ axis
= Moment of Inertia of element δm about OX axis
+ Moment of Inertia of element δm about OY axis

This applies to all the elements of mass in the body

Therefore Moment of Inertia about OZ axis of a laminar in plane OX and OY
= Moment of Inertia of the body about OX axis + Moment of Inertia of the body about OY axis

Axis parallel to axis through the Centre of Gravity

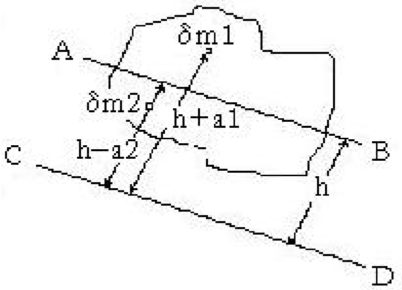


Figure 11: Axis parallel to Axis through the Centre of Gravity

In the diagram,
A–B is an axis throught the Centre of Gravity
C–D is another axis parallel to A–B.
Axis C–D is at distance h from axis A–B..

The Moment of Inertia of $\delta m1$ about C–D = $\delta m_1\,(h + a_1)^2 = \delta m_1\,(h^2 + 2h\, a_1 + a_1^2)$
The Moment of Inertia of $\delta m2$ about C–D = $\delta m_2\,(h - a_2)^2 = \delta m_2\,(h^2 - 2\, h\, a_2 + a_2^2)$

Writing MI for Moment of Inertia
The MI of the body about C–D is
$MI_{CD} = \Sigma\,[\delta m_1\,(h^2 + 2\,h\,a_1 + a_1^2)\,] + \Sigma\,[\delta m_2\,(h^2 - 2\,h\,a_2 + a_2^2)\,]$

$= \Sigma\,[\delta m_1 + \delta m_2\,]h^2 + 2h\,\Sigma\,[\delta m_1\, a_1 - \delta m_2\, a_2] + \Sigma\,[\delta m_1\, a_1^{\,2} + \delta m_2\, a_2^{\,2})$
$= M\,h^2 + 2h$ (Moment of mass about AB) + MI about axis AB

But axis AB passes through the Centre of Gravity therefore Moment of mass about AB = 0

Hence
MI about axis CD = MI about parallel axis through CG + M h ^{2} (A20)

Equations of Motion for Rotational Motion

Consider a small element of mass δm_1 in a flywheel at radius a_1
Apply a Couple C to the flywheel

$$\delta F = \delta m_1\, a_1\;\, d\omega/dt \;\; = \delta m_1\, a_1\, d^2\theta/dt^2$$

But $\quad \delta F\, a_1 \;=\; \delta C$

Therefore $\quad \delta C = \delta m_1\, a_1^{\,2}\, d^2\theta/dt^2$

The angular acceleration of the flywheel, $d^2\,\theta\,/dt^2$, is the same for all elements of mass

$$C = \; d^2\theta/dt^2\, \Sigma\,[\,\delta m_1\, a_1^{\,2}\,] \;\; = I\, d^2\theta/dt^2$$

Thus

$$C = \; I\, d\omega/dt \qquad \text{(A 21)}$$
$$C = \; I\, d^2\theta/dt^2$$

Conservation of Angular Momentum

For linear motion;
Momentum is Mass times Velocity and Change in Momentum is Force times Time.

Similarly
Angular Momentum is Moment of Inertia times Angular Velocity

Angular Momentum = I ω (A22)

And Change in Angular Momentum is Couple times Time

Change in Angular Momentum = C t (A23)

Angular Momentum cannot change unless a Couple is applied.

Hence the Principle of Conservation of Angular Momentum.

Centrifugal and Centripetal Forces.

A mass rotating about an axis exerts a Centrifugal Force on its enclosure. The enclosure exerts a Centripetal Force on the mass.

Centrifugal Forces are outwards, Centripetal Forces are towards the centre

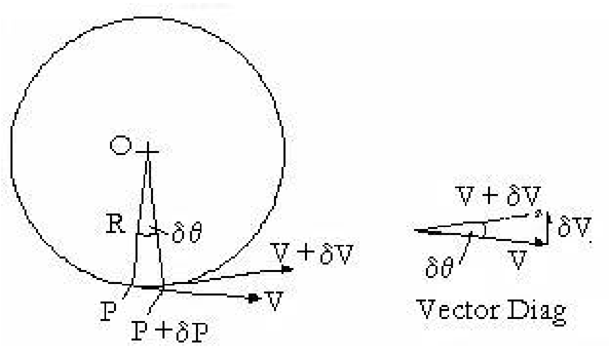


Figure 12: Centrifugal and Centripetal Forces

In time δt, the mass moves through angle δθ from P to P + δP
In time δt, the mass travels a distance V δt = R δθ
Therefore δθ/δt = V/R

The velocity changes from V to (V + δV)

The Vector Diagram of Velocity shows that δV has magnitude V δθ
and is in a direction towards the centre O

But Acceleration is the Change in Velocity in Unit Time
Therefore the Acceleration is V δθ/δt directed towards the centre O

Therefore the Acceleration is V^2/R towards the centre

But $V = R\omega$ where ω is the angular velocity
Hence the Acceleration $= V^2/R = R\,\omega^2$ (A24)

Also Force = Mass times Acceleration
The Centripetal Force acting on the body $= M\,V^2 / R = M\,R\,\omega^2$ (A25)

Change in Moment of Inertia

Consider a Mass M rotating about an axis with angular velocity ω

The Kinetic Energy $= (½)\, I\, \omega^2$
$= (½)\, M\, r^2\, \omega^2$

Let the radius r be increased by δr

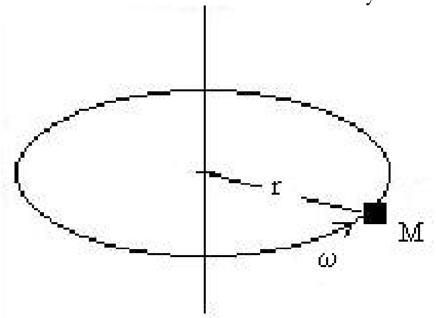


Figure A13: Change in Moment of Inertia

This releases energy due to the Centrifugal Force acting on the Mass (eg this could be used to store energy in a spring)

Energy released $= M\, r\, \omega^2\, \delta r$

By the Principle of Conservation of Energy, this Energy can only come from the Rotational Energy where r changes to $r + \delta r$ and ω changes to $\omega + \delta\omega$

Loss in Kinetic Energy = KE before change − KE after change
Loss in Kinetic Energy $= ½\, M\, r^2\omega^2 - ½\, M\, (r + \delta r)^2\, (\omega + \delta\omega)^2$

Energy Released = Loss in Kinetic Energy
$M\, r\, \omega^2\, \delta r = (½)\, M\, r^2\, \omega^2 - (½)\, M\, (r^2 + 2\, r\, \delta r + \delta r^2)\, (\omega^2 + 2\, \omega\, \delta\omega + \delta\omega^2$
$= (½)\, M\, r^2\, \omega^2 - (½)\, M\, r^2\omega^2 - M\, r\, \omega^2\, \delta r - M\, \omega\, r^2\, \delta\omega)$
+ terms involving products of two small elements

Therefore $M\, \omega\, r^2\, \delta\omega + 2\, M\, r\, \omega^2\, \delta r = 0$
$(1/\omega)\, \delta\omega + (2/r)\, \delta r = 0$
Integrating
$\int (1/\omega)\, d\omega + \int (2/r)\, dr = \text{Constant}$
$\ln(\omega) + 2\ln(r) = \text{constant}$

$\ln(\omega\, r^2)$ = constant
Therefore $(\omega\, r^2)$ = constant
Therefore $(M\, \omega\, r^2)$ = constant

Thus the Principle of Conservation of Angular Momentum is still valid when the Moment of Inertia is changed.. Note that if r is decreased, then ω and the KE are increased

A skater may start spinning with arms outstretched. When the arms are folded, the Moment of Inertia is reduced and therefore the Angular Velocity is increased, ie the skater's speed of spinning accelerates with no apparent additional effort. The Angular Momentum is the same but the Kinetic Energy is increased due to the work done in folding the arms against the centrifugal force.

27 FORCES ACTING ON A BODY

Coplanar Forces

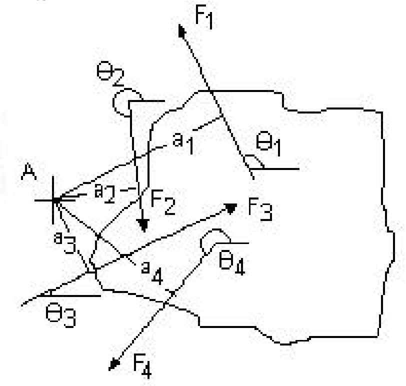


Figure A14: Forces on aa body

Let Forces F_1 , F_2 , F_3 and F_4 be acting in the same plane on a body.

The Forces are equivalent to single Forces R_x and R_y acting horizontally and vertically through an arbitrary point A and a Couple C acting about an arbirary point A

Resolving horizontally

$R_x = F_1 \cos\theta_1 + F_2 \cos\theta_2 + F_3 \cos\theta_3 + F_4 \cos\theta_4$ (A26)

Resolving Vertically

$R_y = F_1 \sin\theta_1 + F_2 \sin\theta_2 + F_3 \sin\theta_3 + F_4 \sin\theta_4$ (A27)

Moments about A

$C = F_1 a_1 - F_2 a_2 + F_3 a_3 - F_4 a_4$ (A28)

Bodies in Equilibrium, all Forces coplanar

If the sum of all the Forces resolved in any direction is not zero, then the body will accelerate in that direction. Thus for equilibrium, the sum of all Forces resolved in any direction = 0

Forces, which are otherwise in equilibrium, may rotate the body.
Thus for equilibrium,
The sum of all the turning moments of the Forces about any point = 0

Thus, when all the forces are in one plane, there are three conditions for equilibrium
i) Sum of the Forces resolved in any one direction in the plane = 0
ii) Sum of the Forces resolved in any other direction in the plane = 0
iii) The sum of the turning Moments of the Forces about any point = 0

If these conditions are met, the sum of the Forces in any other direction in the plane will also be zero. Furthermore the turning moment of the Forces about any other point will also be zero.

The equations for equilibrium can be applied to only part of the body provided the forces within the body are included.

Alternative conditions for equilibrium

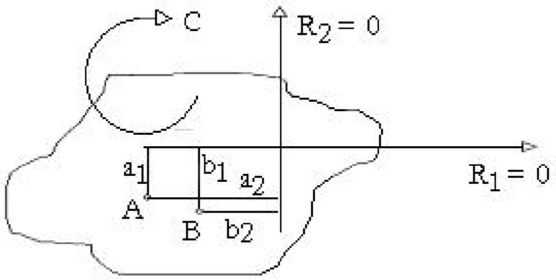


Figure A15: Alternative conditions for equilibrium

1) Suppose the sum of coplanar forces acting on a body are zero resolved in one direction and the total couple about two points A and B are both zero.

The couple about point A = 0 Therefore $C + R_1 a_1 = R_2 a_2$
The couple about point B = 0 Therefore $C + R_1 b_1 = R_2 b_2$
Eliminate C $R_1 (a_1 - b_1) = R_2 (a_2 - b_2)$
But $R_1 = 0$ Therefore $R_2 = 0$ or $a_2 = b_2$
If $a_2 = b_2$ then A and B lie on a line perpendicular to R_1

Thus an alternative set of conditions for equilibrium is;
i) Sum of the Forces resolved in one direction in the plane = 0
ii) The Couple about two points which do not lie on the perpendicular to the direction of the resolved forces are both = 0

2) Suppose the total couple about three points A, B and C are all zero

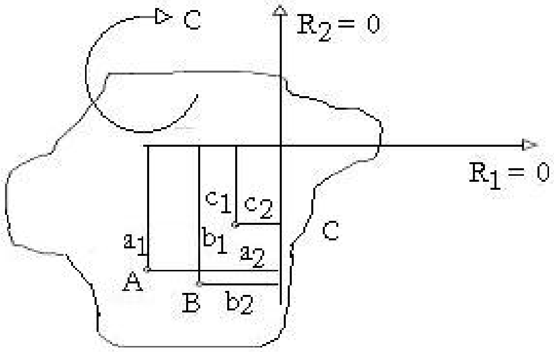


Figure A16: Alternative conditions for equilibrium

The couple about point A = 0 Therefore $C + R_1a_1 = R_2a_2$
The couple about point B = 0 Therefore $C + R_1b_1 = R_2b_2$
The couple about point C = 0 Therefore $C + R_1c_1 = R_2c_2$

Eliminate the couple C $R_1 (a_1 - b_1) = R_2 (a_2 - b_2)$
And $R_1 (a_1 - c_1) = R_2 (a_2 - c_2)$

Therefore
If R_1 and R_2 are not zero
Then $(a_1 - b_1)/(a_2 - b_2) = (a_1 - c_1)/(a_2 - c_2)$
But $(a_1 - b_1)/(a_2 - b_2)$ is the slope of the line from A to B
And $(a_1 - c_1)/(a_2 - c_2)$ is the slope of the line from A to C
Therefore if R_1 and R_2 are not zero, then A, B and C lie in the same line.

If R_1 is zero but R_2 is not zero, then points A, B and C are all on a line perpendicular to R_1 as in alternative (1) above.

If A, B and C do not lie on a straight line, then R_1 and R_2 are both zero

Thus there is a third alternative set of conditions for the body to be in equilibrium.
The total couple about each of three points which do not lie on a straight line are all equal to zero

Bodies in Equilibrium, Forces in three dimensions

When the Forces are in three dimensions, the body is in equilibrium if the Resultants of all the Forces in each of three directions mutually at right angles are all zero and in addition the Resultant Couples about three axes mutually at right angles are also all zero.

Thus the condition for equilibrium is;
The Resultant Forces along arbitrary axes Ox, Oy and Oz are all zero and the Resultant Couples about arbitrary axes Ox', Oy' and Oz' are also all zero.

Three Forces on a body

If only three Forces only act on a body, they must be coplanar for equilibrium. Take moments about the point where two of the Forces cross. The moment of the third Force must be zero, thus it must pass through the same point.

Therefore if three Forces only act on a body in equilibrium, they must be coplanar and either meet at a point or all be parallel.

Examples of bodies in equilibrium

1) Pulley system

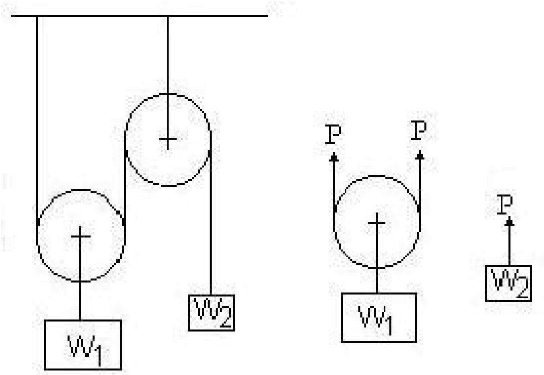


Figure A17: Pulley system

Let the Tension in the rope be P
Consider the weight W_2
For equilibrium, $P = W_2$
Consider the weight W_1
For equilibrium, $2P = W_1$

For equilibrium $W_1 = 2W_2$

2) Lever

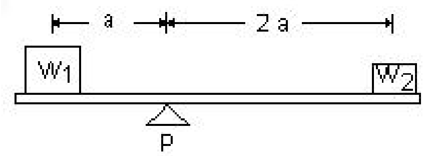


Figure A18: Lever

Resolving Vertically
$P = W_1 + W_2$
Taking Moments about the Fulcrum
$W_1 a = W_2 2a$

For equilibrium $W_1 = 2W_2$

3) Beam

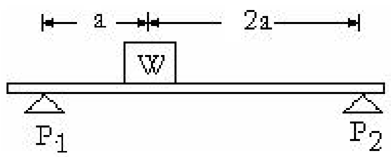


Figure A19: Beam

Take Moments about the left hand support

$P_2\ 3\ a = W\ a$

Therefore $P_2 = W / 3$

Resolving Vertically

$P_1 + P_2 = W$

Therefore $P_1 = W - W / 3$

Therefore $P_1 = 2\ W / 3$

Note P_1 could have been evaluated directly by taking moments about the right hand support

4) Winches connected by Gears

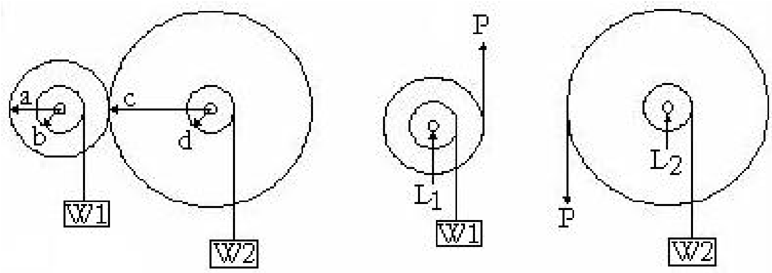


Figure 20: Winches connected by gears

The diagram shows two winches connected by gears.
Let P be the Force on the gear teeth , L_1 and L_2 the Forces on the axles.

Consider the Forces on the left hand gear and winch

Take moments about the axle

$W_1\, b = P\, a$

Resolving Vertically

$L_1 = W_1 - P$

Consider the forces on the right hand gear and winch

$W_2\, d = P\, c$

$L_2 = P + W_2$

Therefore

$W_2\, d/c = P = W_1\, b/a$

$W_2 = W_1\, bc/ad$

5) System with vertical and horizontal Forces

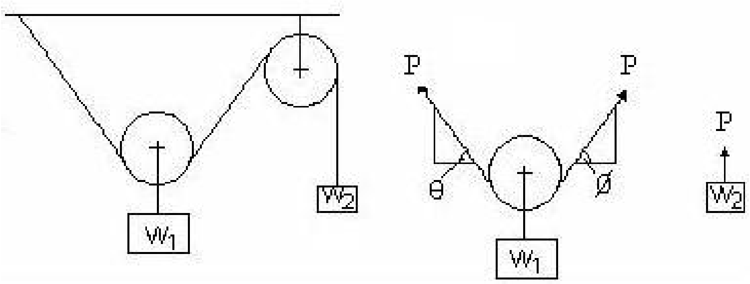


Figure A21: Vertical and Horizontal Forces

Consider the right hand weight $P = W_2$
Consider the left hand pulley
Resolving Horizontally $P \cos\theta = P \cos\phi$
Therefore $\theta = \phi$
Resolving Vertically

$P \sin\theta + P \sin\phi = W_1$

Therefore $W_1 = (2 \sin\theta)\, W_2$

Virtual Work

If the mechanisms are displaced from the equilibrium position, work is done by each of the Forces. When moved a small displacement from the equilibrium position, the total work done is zero, ie work done by some forces is equal and opposite to the work done by the other forces since the Resultant of all the Forces is zero.

This principle could have been used to solve the above examples by equating the work done by each weight when one is displaced a small distance.

Friction

If N is the Force Normal to a surface, the Frictional Force is given by;

$$P = \mu N \qquad \text{(A29)}$$

where μ is the Coefficient of Friction which depends on the properties of the surfaces
μ is small for ice and nearly unity for a rubber tyre on dry tarmac.

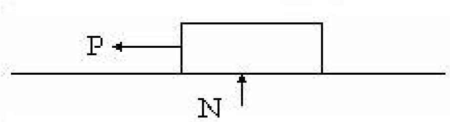


Figure 22: Friction

Where the Normal Force is that due to Gravity, then on a level surface;

$$P = \mu \; M\, g. \qquad \text{(A30)}$$

In practice, it is found that the Coefficient of Friction reduces as soon as the body begins to slide.

The Coefficient of Friction is sometimes quoted as the value with the body sliding and a higher value quoted for the "Coefficient of Stiction" ie the value before sliding occurs.

ABS braking systems are designed to prevent the car tyre sliding and therefore the car stops in a shorter distance.

Capstan

The Capstan has been used on ships for hundreds of years. It consists of a drum that is now driven by a powerful motor (they were powered by a gang of sailors in the past). A sailor loops two or three turns of rope round the drum. When he pulls with a small pull P_1 , a much larger pull P_2 is applied to the rope beyond the capstan. P_2 is directly proportional to P_1 giving the sailor complete control.

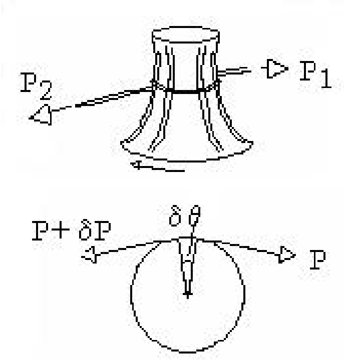


Figure A23: Capstan

Over the small angle $\delta\theta$, the Normal Force = $P \sin \delta\theta$

$\delta\theta$ is small, therefore Normal Force = $P\,\delta\theta$

Therefore Frictional Force $\delta P = \mu P \delta\theta$

Therefore $\delta P/P = \mu\,\delta\theta$

$\text{Log}\,(P_2 / P_1) = \mu\,\theta$

$$P_2 = P_1\, e^{\mu\theta} \qquad \text{(A 31)}$$

Wind Resistance

Experiments show that Wind Resistance is approximately equal to the square of the Velocity times the Frontal Area times a Drag Factor (Cd) which depends on the shape of the object.

$F_W = A\, v^2\, Cd$

28 SIMPLE HARMONIC MOTION (OR SHM)

Basic Equations

An object moves with Simple Harmonic Motion when its acceleration towards the equilibrium position is proportional to its distance from the equilibrium position. The motion is a continuous oscillation.

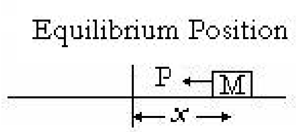


Figure A24: Simple Harmonic Motion

Acceleration is towards the Equilibrium Position and proportional to the distance from it

$P = K\,x$

Therefore $M\,d^2x/dt^2 = -\,K\,x$

$d^2x/dt^2 = -\,(K\,/\,M)\,x$

$d^2x/dt^2 = -(K/M)\,x$ (A32)

This is the basic equation for Simple Harmonic Motion

Multiply by the Integrating Factor $2\,dx/dt$

$2\,dx/dt\; d^2x/dt^2 = -(K/M)\,2\,x\,dx/dt$

Integrate wrt t

$(dx/dt)^2 = -K\,x^2 + \text{Constant}$

$dx/dt = \sqrt{[\text{Constant} - (K/M\,x^2)]}$

$= \sqrt{[(K/M)(a^2 - x^2)]}$

Where the Constant is replaced by another unknown constant $(K/M)\,a^2$

Separate the variables and integrate

$\int [1/\sqrt{(a^2 - x^2)}]\,dx = \sqrt{(K/M)} \int dt$

Hence Arc Sin $(x/a) = \sqrt{(K/M)}\; t + \text{Const}$

$x = a\,[\text{Sin}(\omega\,t + C)]$

where $\omega = \sqrt{(K/M)}$ and C is an arbitrary constant

$x = a\,\text{Sin}(\omega\,t + C)$

The equation for Simple Harmonic Motion is therefore

$x = a\,\text{Sin}(\omega\,t + C)$ (A33)

x varies between $+a$ and $-a$

dx/dt is a maximum when $x = 0$

$dx/dt = 0$ when $x = \pm a$

$x = a\,\text{Sin}(\omega\,t + C)$

$= a\,[\text{Sin}(\omega\,t)\;\text{Cos}\,C + \text{Cos}(\omega\,t)\;\text{Sin}\,C]$

Put $A = a\,\text{Cos}\,C$ and $B = a\,\text{Sin}\,C$

$x = A\,\text{Sin}(\omega\,t) + B\,\text{Cos}(\omega\,t)$ (A34)

The time for one complete oscillation is given by

$\omega\,t = 2\,\pi$

Period for one oscillation $T = 2\,\pi/\omega$

Period $= 2\,\pi\,\sqrt{(M/K)}$ (A35)

If the oscillations have a frequency f then this is the number of oscillations per second.
Therefore the period is $(1/f)$

But the Period $= 2\,\pi/\omega$

Therefore $\omega = 2\,\pi f$

Piston

A piston with a very long connecting rod moves with a motion approaching Simple Harmonic Motion.

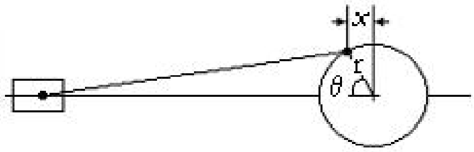


Figure 25: Piston with a long connecting rod

$$x = r \cos \theta$$

If the crankshaft rotates with constant speed ω radians / sec

Then $\theta = \omega t$

Therefore $x = r \cos \omega t$

This is Simple Harmonic Motion

If the connecting rod is long, then the piston movement is closely equal to x

Coil Spring

A spring obeys Hooke' Law the extension is proportional to the tension

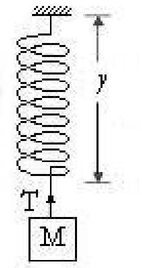


Figure A26: Coil Spring

Let y be the length under tension

L be the unstretched length

A be the length in equilibrium with a mass M attached

T be the tension in the spring

K be the spring constant

$$T = K(y - L)$$

The length is A with a mass M hanging in equilibrium

$$M g = K(A - L)$$

Displace the mass downwards by x from the equilibrium position, length is $y = A + x$

Tension in the spring is given by $T = K(A + x - L)$

Therefore $T - Mg = Kx$

But $(T - Mg)$ is the net force acting on the mass towards the equilibrium position. This force acts on the mass M to reduce x

Therefore

$$M\, d^2x/dt^2 = -(T - Mg) = -Kx$$

$$d^2x/dt^2 = -(K/M)\, x \quad \text{This is the equation for SHM}$$

The time for a complete oscillation is Period $= 2\pi\sqrt{(M/K)}$

Simple Pendulum

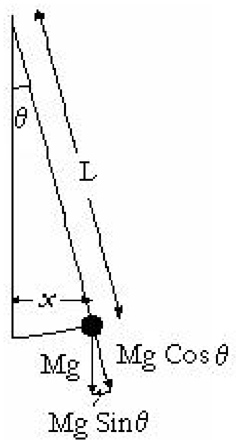


Figure A27: Simple Pendulum

Consider a pendulum length L with mass M all concentrated at the end
Displace by an angle θ from the vertical

Gravitational Force Mg has components
Mg Cos θ down the Pendulum and
Mg Sin θ towards the equilibrium position

If θ is small, Sin θ = θ
and horizontal displacement $x = \mathrm{L}\,\theta$

Force towards the equilibrium position

$$P = (Mg/L)\, x$$

$$M\, d^2x/dt^2 = -(Mg/L)\, x$$

$$d^2x/dt^2 = -(g/L)\, x$$

This is Simple Harmonic Motion

Period for one oscillation = $2\pi/(L/g)$

The pendulum of a grandfather clock has a half period of one second
Therefore $L = g/\pi^2 = 9.81/\pi^2 = 0.994$ metres
The length of the pendulum gives the height of the grandfather clock.

Solid Pendulum

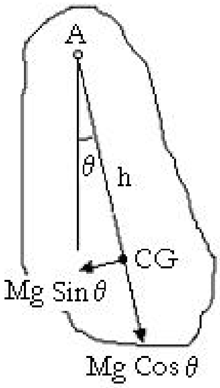


Figure A28: Solid Pendulum

Consider a solid pendulum pivoted at A

The force due to gravity acts through the Centre of Gravity and has components
$Mg \cos \theta$ in a direction away from the pivot and $Mg \sin \theta$ in a direction at right angles.

If h is the distance between the pivot and the CG, then the Couple returning the pendulum to the central position is given by;

$$C = Mgh \sin \theta$$

But if θ is small then $\sin \theta = \theta$

Therefore $I\, d^2\theta/dt^2 = -Mgh\,\theta$
where I is the Moment of Inertia about A

Hence

$$d^2\theta/dt^2 = -(Mgh/I)\,\theta$$ This is Simple Harmonic Motion

$$\text{Period} = 2\pi\sqrt{(I/Mgh)}$$

But $I = M(k^2 + h^2)$ where k is the radius of gyration about the CG

$$\text{Period} = 2\pi\sqrt{[(k^2 + h^2)/(gh)]}$$

Compare with the Simple Pendulum;

Period is the same as a Simple Pendulum length $L = h + k^2/h$

29 STRUCTURES

Pin Jointed Frame

A Pin Jointed Frame consists of a number of bars or tubes or girders each of which is connected to others by joints at each end that are free to rotate. Thus no Couple can be applied to either end. Each member is subjected only to Tension or Compression.

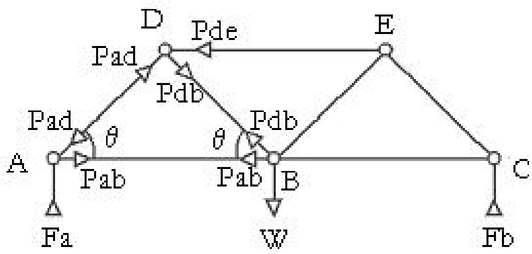


Figure A29: Pin Jointed Frame

The diagram shows a symetrical pin jointed frame carrying a weight W at point B, the centre, and supported at points A and C

Consider the equilibrium of the part of the structure in the vicinity of Point A

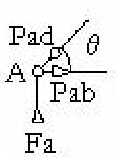


Figure A30: Pin Joint A

By symetry (or by moments about C)

$$Fa = W/2$$

Resolving vertically at Point A

$Pad \sin \theta = Fa$

Therefore $Pad = W/(2 \sin \theta)$

Resolving horizontally at Point A

$Pab = Pad \cos \theta$

Therefore $Pab = W/(2 \tan \theta)$

Resolving vertically at Point D

$Pdb \sin \theta = Pad \sin \theta$

Therefore $Pdb = W/(2 \sin \theta)$

Resolving horizontally at Point D

$Pde = Pad \cos \theta + Pdb \cos \theta$

Therefore $Pde = W/(\tan \theta)$

Members AB and DB are in Tension
Members AD and DE are in Compression

Beams

Beams are solid members.
The Figure shows a beam be firmly fixed into a wall and supporting a weight W at the end.

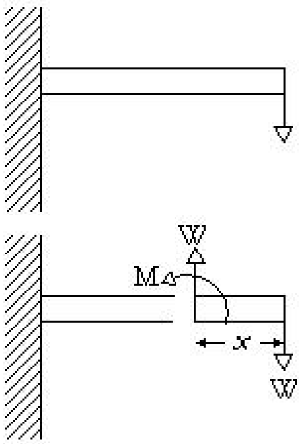


Figure A31: Beam fixed to a wall supporting a weight W at the end

Consider the equilibrium of part of the beam at the outer end and length x

Resolving vertically, there must be a force equal to W acting vertically at the inner end of the part.
The Vertical Force is called the **Shear Force** in the beam.
This Force together with the Weight exert a couple $W x$ on the part.

For equilibrium, this Couple is balanced by horizontal forces in the beam that exert an equal and opposite Couple.

This Couple is called the **Bending Moment** M in the beam at this point. In the diagram,

$M = W x$

Stress and Strain

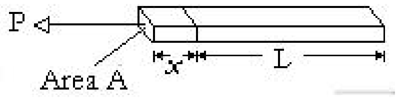


Figure A32: Stress and Strain

Stress is the Force per Unit Area acting on an object.
Strain is the deflection per unit length produced by the Stress

Tensile Stress and Young's Modulus

Let a bar with cross section area A be subjected to a Tensile Force P.

Let this Stress produce an extension x in a bar length L

$$\text{Stress} = P/A \tag{A36}$$
$$\text{Strain} = x/L \tag{A37}$$

For a small Stress, most materials are elastic, ie when the Stress is removed, the bar returns to its former size.

Furthermore for a small Stress,as the Stress is increased, the Strain increases in direct proportion to the Stress.

Thus for a small Tensile Stress, the ratio Stress/Strain is a constant and can be measured. It is called Young's Modulus and denoted by E

$$\text{Young's Modulus} \quad E = \text{Stress / Strain} \tag{A38}$$

If the Force P is at a right angle to the cross section, the Force is a Shear Force. There is similar Modulus for a body subjected to Shear

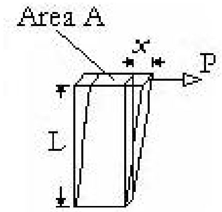


Figure A33: Shear Stress

Shear Stress = P / A
Shear Strain = x / L

Shear Modulus G = Shear Stress / Shear Strain (A39)

If a body is subjected to a large Stress, it can be permanently deformed. The point at which the Stress and Strain first begin to cause a permanent deformation is called the **Elastic Limit.**

Bending Moment in a Beam

When a Beam is subjected to a Bending Moment, it bends a little.

Let the beam be bent to a radius R
Consider a small piece of the beam bent through an angle θ

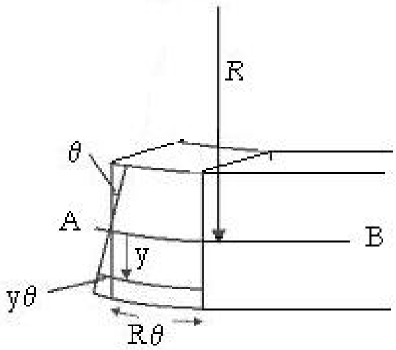


Figure A34: Bending a Beam

Part of the section is in compression and part in tension. The length in the centre of the section is unchanged. This is called the Neutral Axis.

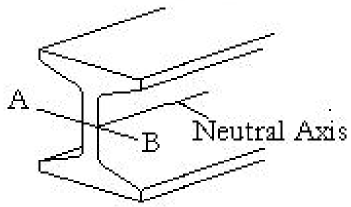


Figure A35: Neutral Axis

At a distance y from the Neutral Axis
Strain = extension / original length
$= y\,\theta \,/\, R\,\theta = y/R$
Hence Stress at y = E times Strain at y = $E\,y/R$

Let p be the Stress at y
Then $p = E\,y/R$
Thus $p/y = E/R$

Consider a small Area δA at a distance y from the Neutral Axis
The Force on this Area = $p\,\delta A$
The Moment of this Force about the Neutral Axis = $p\,y\,\delta A$
Hence $\delta M = [E/R]\,y^2\,\delta A$

Thus the total Bending Moment is

$$M = [E/R]\,\Sigma\,[y^2\,\delta A]$$

$$\text{Put } I = \Sigma\,[y^2\,\delta A] \tag{A40}$$

I is called the Second Moment of Area and is exactly the same as the Moment of Inertia except it has δA instead of δM. It is denoted by I

Thus I can be calculated as for Moment of Inertia about an axis perpendicular to and through the Neutral Axis, axis A-B in the diagram.
Let K be the Radius of Gyration K of the Moment of Inertia of a laminar about this axis.
The Second Moment of Area is then $A K^2$ where A is the total cross sectional area of the beam.
The Second Moment of Area has dimensions L^4.

Total Bending Moment $M = [E/R]\, I$
Thus $E/R = M/I$

Thus for a Beam $p/y = E/R = M/I$ (A41)
This identity is called P Y E R M I

For a Beam with rectangular section depth 2a and width b;

$$M = \int [Eb/R]\, y^2 \;\delta y \quad \text{from } -a \text{ to } +a$$
$$= [E\, b / R]\, y^3/3 \quad \text{from } -a \text{ to } +a$$
$$= 2\, E\, a^3\, b/3\, R$$

Deflections due to Bending Moments

Consider the deflections on a Beam fixed to a wall

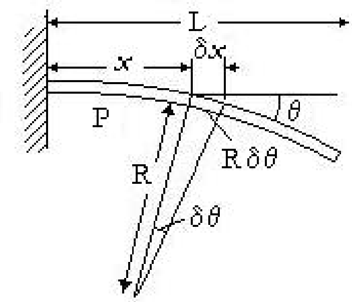


Figure A36: Deflections on a Beam.

Let the Bending Moment be M at a distance x from the wall

The Beam bends with a radius R due to M such that;
$E/R = M/I$

$\delta x = R\, \delta\theta \cos\theta$
But θ is small, therefore $\cos\theta = 1$
$\delta\theta = 1 / (EI)\; M\, \delta x$
$\theta = 1 / (EI) \int M\, dx$ from 0 to x

Vertical deflection $\delta d = \text{Tan}\,\theta\,\delta x$

But θ is small, therefore $\text{Tan}\,\theta = \theta$

$$\delta d = \theta\,\delta x$$

At the end of the Beam, $d = \int \theta\, dx$ from 0 to L

Beam with Weight W at end

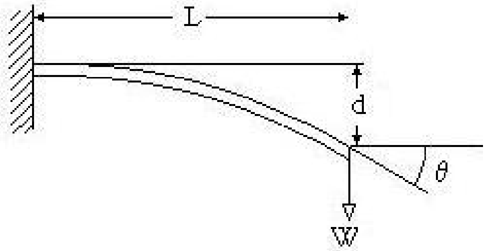


Figure A37: Beam Weight W at end

Bending Moment x from the wall M = W (L − x)
At x from the wall θ = 1/(EI) ∫ W (L − x) dx from 0 to x
= 1/(EI) [WLx − x^2/2]
Therefore d = 1/(EI) ∫ [WLx − x^2/2] dx from 0 to L
d = 1/(EI) [WL x^2/2 − x^3/6] from 0 to L = WL^3/ (3EI)
At L from the wall θ = 1/(EI) [WL^2 − L^2/2] = WL^2/(2EI)

Beam with Weight at end

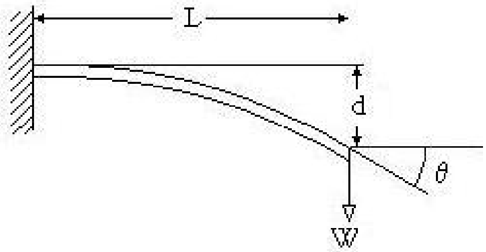


Figure A38: Beam Weight W at end

$$d = WL^3/(3EI) \qquad (A42)$$
$$\theta = WL^2/(2EI) \qquad (A43)$$

Beam with Weight distributed over Beam

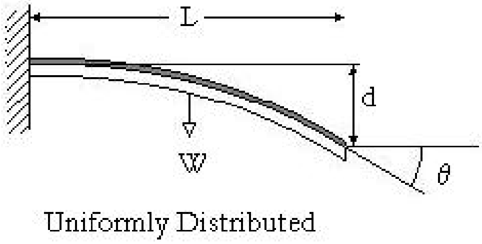


Figure A39: Uniformly Distributed Load

At x from the wall, $M = W(L - x)^2 / L$ and by a similar method

$$d = WL^3 / (8EI) \tag{A44}$$
$$\theta = WL^2/(6EI) \tag{A45}$$

Beam with Bending Moment at the end

The Bending Moment is constant at M throughout the Beam

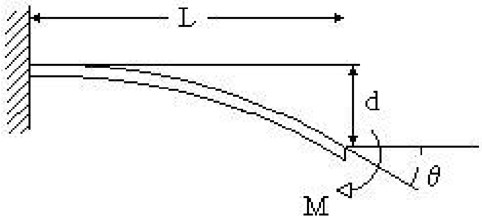


Figure A40: Bending Moment

$$d = M L^2 / (2EI) \tag{A46}$$
$$\theta = M L/(EI) \tag{A47}$$

These identities can be used to calculate the deflections in a large range of structures.

Example
Calculate the deflection under a point load on a portal frame, height H and width B

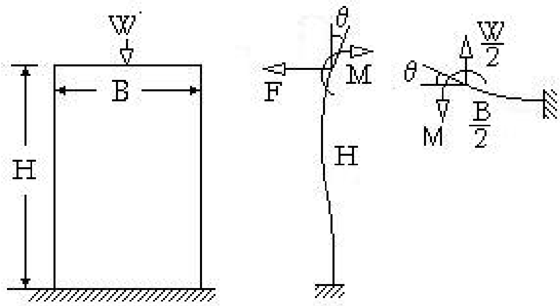


Figure A41: Portal Frame

Horizontal deflection of the verticals $= 0$

Therefore $F H^3 / (3EI) = M H^2 / (2EI)$

Therefore $F = 3 M / (2 H)$

$$\theta = M H / (EI) - F H^2 / (2EI) = M H / (EI) [1 - 3/4]$$
$$= M H / (4 EI)$$

The deflection of the top member $= \theta$

$$\theta = (W/2) (B/2)^2 / (2EI) - M (B/2) / (EI)$$

Equate the values for θ and multiply by EI

$$M H / 4 = W B^2/16 - M B/2$$
$$4 M H = W B^2 - 8 M B$$
$$M (4H + 8B) = W B^2$$

The deflection of the top member

$$d = (W/2) (B/2)^3 / (3EI) - M (B/2)^2 / (2EI)$$
$$d = W B^3 / (48 EI) - W B^2 (B/2)^2 / [(4H + 8B) (2EI)]$$
$$d = W B^3 [\{1/(48EI)\} - B / \{8EI (4H + 8B)\}$$
$$d = [W B^3 / (16EI)] [(1/3) - B/(2H + 4B)]$$
$$d = [W B^3 / (96EI)] [(2H + B)/(H + 2B)]$$

Torque in a solid Bar, radius a

When a Couple is applied to a bar, it twists. The angle of the deflection is proportional to the Couple.

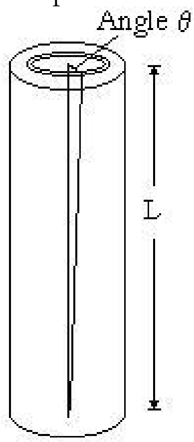


Figure A42: Torque in a solid bar

Consider a thin tube, length L, concentric with the centre of the bar, radius r and thickness δr

Apply a Couple δM to the tube
Let the angular twist in the tube be θ.

The thin tube is in Shear
Cross sectional Area $= 2 \pi r \delta r$

Force on the thin tube $= \delta M / r$
Shear Stress $= \delta M / [2 \pi r \delta r\, r] = \delta M / [2 \pi r^2 \delta r]$
Shear Strain $= r \theta / L$

But Shear Stress / Shear Strain $= G$

$$\delta M / 2\pi r^2 \delta r = G\, r \theta / L$$

Thus $\delta M = 2 \pi r^2 \delta r\, G\, r \theta / L$

$$= [2\pi G \theta / L]\, r^3 \delta r$$

Therefore;
$M = \int [2\pi G \theta / L]\, r^3 \delta r$ from 0 to a, where a is the radius of the bar
$M = [2\pi G \theta / L]\, a^4/4$

Therefore $M = [\pi G a^4 / (2 L)]\, \theta$
Or $\theta = 2 M L / [\pi G a^4]$ (A48)

Stress in a pipeline

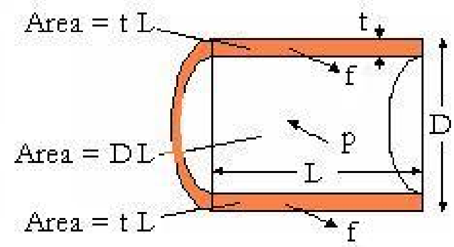


Figure A43: Hoop Stress

p D L = 2 f t L
Hoop stress f = p D / 2 t

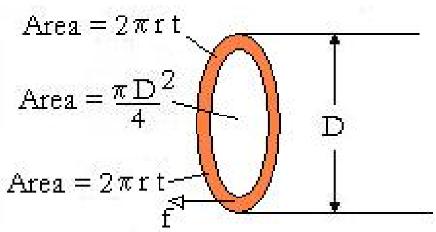


Figure 44A: Longitudinal Stress
$p \, \pi \, D^2 / 4 = f \, \pi \, D \, t$
Longitudinal stress f = p D / 4

where p is the pressure, D is the outside diameter and t is the wall thickness

The outside diameter is used as the pressure acting on the inside diameter squeezes the wall increasing the stress.

30 HANGING CHAINS

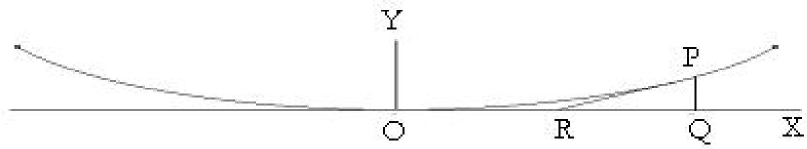


Figure A45: Suspension Bridge

Suspension Bridge
Let P be a point on the Suspension Bridge chain at (x,y).
Let the roadway be fully supported by the chain with weight uniformly distributed along the OX axis at w per unit length. Weight of OQ = w x
Consider the equilibrium of the bridge between points O, P and Q.
The Weight of the Roadway acts downwards at the mid point of OQ
The Force in the Chain at P acts tangentially along the chain
The Force in the Chain at O acts horizontally along the chain
For equilibrium, these three forces meet at a point, ie they meet at R such that RP is tangential to the chain and R is the mid point of OQ.

Therefore at point P $dy/dx = 2y/x$ and $\int dy/y = 2\int dx/x$
Integrating $\ln(y) = 2\ln(x) + \text{const}$
Therefore the curve is a parabola $y = A x^2$

Let angle PRQ be θ. Then dy/dx at point P is $\tan\theta$.
Let the Force in the chain at P be F and at O be F_0
For equilibrium
Resolving horizontally $F \cos\theta = F_0$
Resolving vertically $w x = F \sin\theta$.
Therefore $\tan\theta = dy/dx = w x / F_0$
Integrate $\int dy = \int [w/F_0]\, x\, dx + c$
$y = [w/(2F_0)]\, x^2 + c$
With axes chosen as shown, $y = 0$ when $x = 0$, therefore $c = 0$
Thus the Chain is parabolic following the curve

$$y = w x^2 / (2F_0) \qquad \text{(A49)}$$

$F = w x / \sin\theta = [w x / \tan\theta]\sqrt{(1 + \tan^2\theta)}$
$\tan\theta = dy/dx = w x / F_0$
At point P, $F = F_0\sqrt{[1 + (w x / F_0)^2]} = \sqrt{[F_0^2 + (w x)^2]}$ (A50)

Catenary

A heavy chain on its own hangs in a curve called a Catenary.

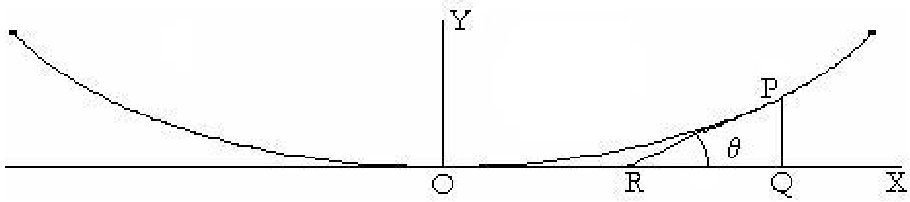


Figure A46: Catenary

Let angle PRQ be θ. Then dy/dx at point P is $\tan\theta$..
Let the Force in the chain at P be F and at O be F_0
Let the weight of the chain be w per unit length

Consider the length OP of the chain
Weight = w s where s = length of the arc OP
Resolving horizontally $F_0 = F \cos\theta$
Resolving vertically $w s = F \sin\theta$
Therefore $\tan\theta = w s / F_0$
Put $F_0 / w = c$
Then $s = c\tan\theta = c\, dy/dx$ (A51)
This is the basic equation for a Catenary

Differentiate $ds/dx = c\, d^2y/dx^2$
But $\delta s^2 = \delta x^2 + \delta y^2$
Therefore $\delta s/\delta x = \sqrt{[1 + (\delta y/\delta x)^2]}$

$$ds/dx = \sqrt{[1 + (dy/dx)^2]}$$

Put $dy/dx = u$ $\qquad$ $c\, du/dx = \sqrt{(1 + u^2)}$

Integrate $\qquad c \int [1/\sqrt{(1 + u^2)}]\, du = x + \text{const}$

$\qquad c \text{ Arc Sinh } (u) = x + \text{const}$

Thus $\qquad u = \text{Sinh } [(x + \text{const}) / c]$

The origin O is at the lowest point of the chain

Therefore $u = dy/dx = 0$ when $x = 0$ therefore const $= 0$

Hence $\qquad dy/dx = \text{Sinh } (x / c)$

Integrate $\qquad y = c \text{ Cosh } (x / c) + \text{const}$

When $x = 0$, $\underline{y} = 0$ therefore const $= -c$

Equation for a Catenary with the Origin at the lowest point is;

$$y = c\,[\text{Cosh } (x / c) - 1] \qquad \text{(A52)}$$

31 GYROSCOPES

Characteristics of gyroscopes

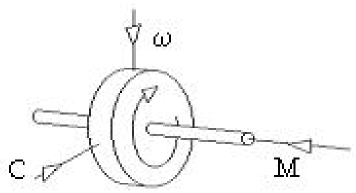


Figure A47: Gyroscope

Let a Flywheel spin with Angular Momentum M shown as a Vector in a corkscrew direction.

Apply a Couple C again shown as a Vector in a corkscrew direction.

The gyroscope will then rotate with a constant Angular Velocity ω again shown as a Vector in the corkscrew direction.

M, C and ω are related by the equation

$$C = M \omega$$

This equation is usually written as the Vector Cross Product

$$\mathbf{C} = \omega \times \mathbf{M} \qquad \text{(A53)}$$

This shows the corkscrew direction of Vector **C** is obtained by rotating from Vector ω to Vector **M**

The Vectors must of course be in corresponding Units
For example in the MKS system

M in Kilogram Metre^2
ω in Radians per second
C in Newton Metres

INDEX

E

F

G

H

I

K

L

M

N

O

P