The McGraw·Hill Companies

SUPPLEMENTS

Log on to http://www.mhhe.com/vohra3e

Mathematics and Statistics Refresher

1. Matrix Algebra
2. Differential Calculus
3. Theory of Probability
4. Probability Distributions

Summary of the chapters of the book

Statistical and other tables

Special offer for Instructors !!

QUANTITATIVE TECHNIQUES IN Management

THIRD EDITION

N D VOHRA

Tata McGraw-Hill

Published by the Tata McGraw-Hill Publishing Company Limited,
7 West Patel Nagar, New Delhi 110 008.

Second reprint 2007
RYXQCRDYRBQBZ

ISBN 0-07-061193-9

Head : Higher Education & School: S. Raghothaman
Executive Publisher: V. Biju Kumar
Sponsoring Editor: Tapas K. Maji
Senior Copyeditor: Anubha Srivastava

Deputy General Manager-Higher Education and Sales: Michael J. Cruz
Asst. Product Manager : Sunil K Agarwal

AGM : Production : B.L. Dogra
Manager, Production : P L Pandita

Typeset at Script Makers,19, A1-B, DDA Market, Pashchim Vihar, New Delhi 110 063, and printed at Pashupati Printers Pvt Ltd., 429/16, Gali No.1, Friends Colony, Shahdara, Delhi 110095

Cover Printer: Rashtriya Printers

The McGraw-Hill Companies

Contents

Preface to the Third Edition vii
Preface to the First Edition ix

1. Decision-Making and Quantitative Techniques **3**
1.1 Introduction *3*
1.2 Quantitative Approach to Decision Making: OR/MS *4*
1.3 Quantitative Analysis and Computer-based Information Systems *12*
1.4 Plan of the Book *13*
Key Points to Remember 14
Test Your Understanding 15
Exercises 15

2. Linear Programming I: Formulation and Graphic Solution **21**
2.1 Introduction *21*
2.2 Linear Programming *22*
2.3 Formulation of Linear Programming Problems *22*
2.4 General Statement of Linear Programming Problems *24*
2.5 Assumptions Underlying Linear Programming *25*
2.6 Solution to Linear Programming Problems—Graphic Method *26*
2.7 Some Special Cases *33*
Review Illustrations 38
Key Points to Remember 56
Test Your Understanding 57
Exercises 58
Practical Problems 59

3. Linear Programming II: Simplex Method **75**
3.1 Introduction *75*
3.2 Simplex Method *75*
3.3 Solution to Maximisation Problems *77*
3.4 Solution to Minimisation Problems *89*
3.5 Big-M Method *90*
3.6 Two-Phase Method *94*
3.7 Some Special Topics *98*
Review Illustrations 111
Key Points to Remember 128
Test Your Understanding 128
Exercises 130
Practical Problems 131

4. Linear Programming III: Duality and Sensitivity Analysis **143**
4.1 Introduction *143*
4.2 Duality in Linear Programming *143*
4.3 Economic Interpretation of Dual *152*
4.4 Sensitivity Analysis *158*
Review Illustrations 172
Key Points to Remember 186
Test Your Understanding 187
Exercises 188
Practical Problems 189

5. Specially Structured Linear Programmes I: Transportation and Transhipment Problems **203**
5.1 Introduction *203*
5.2 Problem Statement *204*
5.3 Solution to the Transportation Problem *206*
5.4 Some Special Topics *219*
5.5 Dual of the Transportation Model *230*
5.6 Production Scheduling and Inventory Control *232*
5.7 Transhipment Problem *235*
Review Illustrations 239
Key Points to Remember 256
Test Your Understanding 257
Exercises 258
Practical Problems 259

6. Specially Structured Linear Programmes II: Assignment Problem **275**
6.1 Introduction *275*
6.2 Assignment Problem: A Variant of the Transportation Problem *276*
6.3 Solution to the Assignment Problem *277*
6.4 Some Special Cases *285*
6.5 Dual of the Assignment Problem *292*
Review Illustrations 295
Key Points to Remember 309
Test Your Understanding 309
Exercises 310
Practical Problems 311

7. Extensions of Linear Programming: Integer Programming and Goal Programming **323**
7.1 Introduction *323*
7.2 Integer Programming *323*
7.3 Solution to IPPs *332*
7.4 Goal Programming *349*
7.5 Modified Simplex Method for Goal Programming *362*
Review Illustrations 365
Key Points to Remember 386
Test Your Understanding 387
Exercises 388
Practical Problems 388

8. Sequencing 399
8.1 Introduction 399
8.2 The Sequencing Problem 400
8.3 Solution to Sequencing Problems 401
8.4 Maintenance Crew Schéduling—An Application 413
Review Illustrations 414
Key Points to Remember 417
Test Your Understanding 418
Exercises 419
Practical Problems 419

9. Inventory Management 427
9.1 Introduction 427
9.2 Types of Inventory 427
9.3 Inventory Management Systems 430
9.4 Fixed Order Quantity System 430
9.5 The Question of Safety Stock 447
9.6 Periodic Review System 458
9.7 Ss System 459
9.8 One-period Model of Inventory Management 460
9.9 Selective Approaches to Inventory Control 462
Review Illustrations 466
Key Points to Remember 481
Test Your Understanding 482
Exercises 484
Practical Problems 485

Appendix 9A 499
9A.1 Determination of EOQ(Q^*) for the Classical EOQ Model 499
9A.2 Determination of the Optimal Lot Size Q^* for the Production-Lot Size Model 500
9A.3 Determination of Q^* and S^* for the Planned Shortages Model 500

10. Queuing Theory 505
10.1 Introduction 505
10.2 General Structure of Queuing System 506
10.3 Operating Characteristics of Queuing System 510
10.4 Queuing Models 510
Review Illustrations 530
Key Points to Remember 542
Test Your Understanding 543
Exercises 544
Practical Problems 545

11. Replacement Theory 555
11.1 Introduction 555
11.2 Replacement Policy for Equipment which Deteriorates Gradually 556
11.3 Replacement of Items that Fail Suddenly 561
11.4 Staff Replacement 566
Review Illustrations 567

Key Points to Remember *577*
Test Your Understanding *578*
Exercises *579*
Practical Problems *579*

Appendix 11A 587

12. PERT and CPM **593**
12.1 Introduction *593*
12.2 PERT/CPM Networks *595*
12.3 Network Analysis *602*
12.4 Resource Analysis and Allocation *610*
12.5 Programme Evaluation and Review Technique (PERT) *623*
12.6 Difference between PERT and CPM *633*
Review Illustrations *633*
Test Your Understanding *655*
Key Points to Remember *658*
Exercises *659*
Practical Problems *660*

13. Decision Theory **681**
13.1 Introduction *681*
13.2 One-stage Decision Making Problems *682*
13.3 Multi-stage Decision Making Problems: Decision Tree *697*
13.4 Utility Theory: Utility as Basis for Decision-Making *701*
Review Illustrations *706*
Key Points to Remember *726*
Test Your Understanding *726*
Exercises *727*
Practical Problems *728*

14. Markov Chains **747**
14.1 Introduction *747*
14.2 Brand Switching Example *747*
14.3 Markov Processes *749*
14.4 Markov Analysis: Input and Output *750*
Review Illustrations *760*
Key Points to Remember *769*
Test Your Understanding *770*
Exercises *771*
Practical Problems *772*

15. Theory of Games **781**
15.1 Introduction *781*
15.2 Game Models *782*
15.3 Two-Person Zero-Sum Games and Their Solution *782*
15.4 Solution of $2 \times n$ and $m \times 2$ Games *791*
15.5 Solution of $m \times n$ Games—Formulation and Solution as an LPP *796*
15.6 Limitations of the Game Theory *801*

Review Illustrations 802
Key Points to Remember 812
Test Your Understanding 813
Exercises 814
Practical Problems 815

16. Dynamic Programming **827**
16.1 Introduction 827
16.2 Dynamic Programming versus Linear Programming 828
16.3 Dynamic Programming: A Network Example 828
16.4 Terminology 832
16.5 Deterministic Versus Probabilistic Dynamic Programming 835
Review Illustrations 843
Key Points to Remember 852
Test Your Understanding 853
Exercises 854
Practical Problems 854

17. Simulation **863**
17.1 Introduction 863
17.2 Process of Simulation 864
17.3 Monte Carlo Simulation 865
17.4 Simulation of an Inventory System 867
17.5 Simulation of Queuing System 870
17.6 Advantages and Disadvantages of Simulation 874
17.7 Applications of Simulation 874
Review Illustrations 874
Key Points to Remember 890
Test Your Understanding 890
Exercises 891
Practical Problems 891

18. Investment Analysis and Break-Even Analysis **903**
18.1 Introduction 903
18.2 Time Value of Money 903
18.3 Annuities 905
18.4 Investment Analysis: Capital Budgeting 909
18.5 Methods of Incorporating Risk into Capital Budgeting 916
18.6 Break-Even Analysis 934
Review Illustrations 942
Key Points to Remember 954
Test Your Understanding 954
Exercises 956
Practical Problems 957

19. Forecasting **969**
19.1 Introduction 969
19.2 Forecasting Models 969
19.3 Qualitative Models of Forecasting 970

19.4 Time Series Models of Forecasting *971*
19.5 Causal Models of Forecasting *982*
Review Illustrations 988
Key Points to Remember 990
Test Your Understanding 991
Exercises 992
Practical Problems 992

Appendix A

Key to 'Test Your Understanding' 997

Appendix B

Statistical and Other Tables 1001

Appendix C

Answers to Practical Problems 1015

Bibliography 1042

Index 1045

Chapter 1

Decision-Making and Quantitative Techniques

Chapter Overview

This book is about the use of quantitative techniques in managerial decision making. Broadly speaking, decision making involves choosing a course of action from various available alternatives. The job of a manager, in the process of selecting from among available alternatives, is facilitated in a large measure by the application of appropriate quantitative techniques when, and to the extent, a problem is quantified. This approach to decision making is known by several names like operations research, operational research, management science, quantitative analysis, etc.

This introductory chapter gives some details about the decision making process and an idea about the nature and methodology of the quantitative analysis. Finally, a plan of the book is presented which contains a brief account of the contents of each of the chapters to follow.

Chapter 1

Decision-Making and Quantitative Techniques

1.1 INTRODUCTION

Decision-making is an essential part of the management process. Although authorities differ in their definitions of the basic functions of management, everybody agrees that one is not a manager unless he or she has some authority to plan, organise and control the activities of an enterprise and behaviour of others. Thus, decision-making pervades the activities of every business manager. Further, since to carry out the key managerial functions of planning, organising, directing and controlling, the management is engaged in a continuous process of decision-making pertaining to each of them, we can go to the extent of saying that management may be regarded as equivalent to decision-making.

Traditionally, decision-making has been considered purely as an art, a talent that is acquired over a period of time through experience. It has been considered so because a variety of individual styles can be observed in the handling and successful solving of similar managerial problems by different people in actual business. However, the environment in which the management has to operate nowadays is complex and fast changing. There is a greater need for supplementing the art of decision-making by systematic and scientific methods. A systematic approach to decision-making is necessary because today's business and the environment in which it functions are far more complex than in the past, and the cost of making errors may be too high. Most of the business decisions cannot be made simply on the basis of rule of thumb, using common sense and/or 'snap' judgement. Common sense may be misleading and snap judgements may have painful implications. For large businesses, a single wrong decision may not only be ruinous but may also have ramifications in national economy. As such, the present-day management cannot rely solely on a trial-and-error approach and the managers have to be more sophisticated. They should employ scientific methods to help them make proper choices. Thus, the decision-makers in the business world today must understand the scientific methodology of making decisions. This calls for (1) defining the problem in a clear manner, (2) collecting pertinent facts, (3) analysing the facts thoroughly, and (4) deriving and implementing the solution.

1.2 QUANTITATIVE APPROACH TO DECISION MAKING: OR/MS

Managerial decision-making is a process by which the management, when faced with a problem, chooses a specific course of action from a set of possible options. In making a decision, a business manager attempts to choose the most effective course of action in the given circumstances in attaining the goals of the organisation. The various types of decision-making situations that a manager might encounter can be listed as follows:

1. Decisions under *certainty,* where all facts are known fully and for sure, or under *uncertainty* where the event that would actually occur is not known but probabilities can be assigned to various possible occurrences.
2. Decisions for one time period only, called *static* decisions, or a sequence of interrelated decisions made either simultaneously or over several time periods, called *dynamic* decisions.
3. Decisions where the opponent is *nature* (digging an oil well, for example) or a *rational opponent* (for instance, setting the advertising strategy when the actions of competitors have to be considered).

These classes of decision-making situations are not mutually exclusive and a given situation would exhibit characteristics from each class. Stocking of an item for sale in a certain trade fair, for instance, illustrates a static decision-making situation where uncertainty exists and nature is the opponent.

The elements of any decision are:

(a) a decision-maker, who could be an individual, group, organisation, or society;
(b) a set of possible actions that may be taken to solve the decision problem;
(c) a set of possible states that might occur;
(d) a set of consequences (pay-off) associated with various combinations of courses of action and the states that may occur; and
(e) the relationship between pay-off and the values of the decision-maker.

In an actual decision-making situation, the definition and identification of alternatives, the states and the consequences, are most difficult, albeit not the most crucial aspects of the decision problem.

In real life, some decision-making situations are simple while others are not. Complexities in decision situations arise due to several factors. These include the complicated manner of interaction of the economic, political, technological, environmental and competitive forces in society, the limited resources of an organisation, the values, risk attitudes and knowledge of the decision-makers and the like. For example, a company's decision to introduce a new product will be influenced by such considerations as market conditions, labour rates and availability, and investment requirements and availability of funds. The decision will be of multidimensional response, including the production methodology, cost and quality of the product, price, package design, and marketing and advertising strategy. The result of the decision would conceivably affect every segment of the organisation. The essential idea of the quantitative approach to decision-making is that if the factors that influence the decisions can be identified and quantified, it becomes easier to resolve the complexity of the tools of quantitative analysis. In fact, a large number of business problems have been given a quantitative representation with varying degrees of success and it has led to a general approach which is variedly designated as *operations research* (or *operational research*), *management science, systems analysis, decision analysis, decision science* and so on. Quantitative analysis is now extended to several areas of business operations and represents probably the most effective approach to handling of some types of decision problems.

A significant benefit of attaining some degree of proficiency with quantitative methods is exhibited in the way problems are perceived and formulated. A problem has to be well defined before it can be formulated into a well-structured framework for solution. This requires an orderly and organised way of thinking.

Two observations may be made here. First, it should be understood clearly that, although quantitative analysis represents a scientific approach to decision-making, a decision by itself does not become a good and right decision for adoption merely because it is made within an orderly and mathematically precise framework. Quantification at best is an aid to business judgement and not its substitute. A certain degree of constructive skepticism is as desirable in considering a quantitative analysis of business decisions as it is in any other process of decision-making. Further, some allowances should be made for qualitative factors involving morale, motivation, leadership and so on, which cannot be ignored. But they should not be allowed to dominate to such an extent that the quantitative analysis seems to be an interesting, but worthless, academic exercise. In fact, the manager should seek some balance between quantitative and qualitative factors.

Secondly, it may be noted that various names for quantitative analysis—operations research, management science and so on—connote more or less the same general approach. We shall not attempt to discuss the differences among various labels, as it is prone to create more heat than light, but only state that the basic reason for so many titles is that the field is relatively new, and there is no consensus regarding which field of knowledge it includes. In this book, we use the terms *management science, operations research*, and *quantitative analysis* interchangeably.

We shall now briefly discuss operations research—its historical development, nature and characteristics, and methodology.

1.2.1 Historical Development of Operations Research (OR)

While it is difficult to mark the 'beginning' of the operations research/management science, the scientific approach to management can be traced back to the era of Industrial Revolution and even to periods before that. But operations research, as it exists today, was born during the second world war when the British military management called upon a group of scientists to examine the strategies and tactics of various military operations, with the intention of efficient allocation of scarce resources for the war effort. During this period, many new scientific and quantitative techniques were developed to assist military operations. The name *operational research* was derived directly from the context in which it was used—research activity on operational areas of the armed forces. British scientists spurred the American military management to similar research activities (where it came to be known as operations research). Among the investigations carried out by them were included the determination of (i) optimum convoy size to minimise losses from submarine attacks, (ii) the optimal way to deploy radar units in order to maximise potential coverage against possible enemy attacks, and (iii) the invention of new flight patterns, and the determination of the correct colour of the aircraft in order to minimise the chance of detection by the submarines.

After the war, operations research was adopted by the industry and some of the techniques that had been applied to the complex problems of war were successfully transferred and assimilated for use in the industrialised sector.

The dramatic development and refinement of the techniques of operations research and the advent of digital computers are the two prime factors that have contributed to the growth and application of OR in the post-war period. In the 1950s, OR was mainly used to handle management problems that were clear-cut, well-structured and repetitive in nature. Typically, they were of a tactical and operational nature, such as inventory control, resource allocation, scheduling of construction projects and so on. Since the 1960s, however, formal approaches have been increasingly adopted for the less well-structured planning problems as well. These problems are strategic in nature and are the ones that affect the future of the organisation. The development of corporate planning models and those relating to the financial aspects of the business, for example, are such type of problems. Thus, in the field of business and industry, operations research helps the management determine their tactical and strategic decisions more scientifically.

1.2.2 Nature and Characteristic Features of OR

In general terms, we can regard operations research as being the application of scientific methods to decision making. Thus, operations research attempts to provide a systematic and rational approach to the fundamental problems involved in the control of systems by making decisions which, in a sense, achieve the best results considering all the information that can be profitably used. A classical definition of OR is given by Churchman et al, "... Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of operations with optimum solutions to the problems."* Thus, it may be regarded as the scientific method employed for problem solving and decision-making by the management.

The significant features of operations research are given below.

1. Decision-making Primarily, OR is addressed to managerial decision-making or problem solving. A major premise of OR is that decision-making, irrespective of the situation involved, can be considered a general systematic process that consists of the following steps:

(a) Define the problem and establish the criterion, which will be used. The criterion may be the maximisation of profits, utility and minimisation of costs, and so on.
(b) Select the alternative courses of action for consideration.
(c) Determine the model to be used and the values of the parameters of the process.
(d) Evaluate the alternatives and choose the one that is optimal.

2. Scientific Approach OR employs scientific methods for the purpose of solving problems, and there is no place of whims and guesswork in it. It is a formalised process of reasoning and consists of the following steps:

(a) The problem to be analysed is defined clearly and the conditions for observations are determined.
(b) Observations are made under varying conditions to determine the behaviour of the system.
(c) On the basis of the observations, a hypothesis describing how the various factors involved, are believed to interact and the best solution to the problem is formulated.
(d) To test the hypothesis, an experiment is designed and executed. Observations are made and measurements are recorded.
(e) Finally, the results of the experiments are analysed and the hypothesis is either accepted or rejected. If the hypothesis is accepted, the best solution to the problem is obtained.

3. Objective OR attempts to locate the best or optimal solution to the problem under consideration. For this purpose, it is necessary that a measure of effectiveness be defined, which is based on the goals of the organisation. This measure is then used as the basis to compare the alternative courses of action.

---

* Churchman, C.W., Ackoff, R.L. and Acnoff, *Introduction to Operations Research*, John Wiley & Sons, New York, 1977, p.9.

4. Inter-disciplinary Team Approach OR is inter-disciplinary in nature and requires a team approach to a solution to the problem. No single individual can have a thorough knowledge of the myriad aspects of operations research and how the problems may be addressed. Managerial problems have economic, physical, psychological, biological, sociological and engineering aspects. This requires a blend of people with expertise in the areas of mathematics, statistics, engineering, economics, management, computer science, and so on. Of course, it is not always so. Some problem situations may be adequately handled even by one individual.

5. Digital Computer Use of a digital computer has become an integral part of the operations research approach to decision-making. The computer may be required due to the complexity of the model, volume of data required or the computations to be made. Many quantitative techniques are available in the form of 'canned' programmes.

1.2.3 Methodology of Operations Research

The basic and dominant characteristic feature of operations research is that it employs mathematical representations or models to analyse problems. This distinctive approach represents an adaptation of the scientific methodology used by other physical sciences. The scientific method translates a real given problem into a mathematical representation, which is solved and re-transformed into the original context. The OR approach to problem solving consists of the following steps.

1. Formulate the problem.
2. Determine the assumptions (model building) and formulate the problem in a mathematical framework.
3. Acquire the input data.
4. Solve the model formulated and interpret the results.
5. Validate the model.
6. Implement the solution obtained.

However, one step need not be completed fully before the next is started. In fact, one or more of the steps may be modified to some extent before final results are implemented. This would of course, necessitate the subsequent steps being modified.

The steps are shown in Fig. 1.1 and we now discuss each of these one by one.

1. Problem Formulation The first step in quantitative analysis is to develop a clear and concise statement of the problem. In many cases, defining the problem proves to be the most important and difficult step. It is essential, therefore, that the root problem should be identified and understood in the first place. Logically speaking, we cannot expect to get the right answer if the problem is identified incorrectly. In that case, the solution derived from it is apt to be useless and all the efforts in that direction shall be a waste. The problem should be identified properly because often what is described as a problem may only be its symptom. For example, excessive costs *per se* do not constitute a problem. They are only an indication of some problem that may, for instance, be improper inventory levels, excessive wastage, and the like. Often the symptoms of a problem may extend beyond a single manager's control to other personnel and other departments in an organisation. It is essential, therefore, to go beyond symptoms of the problem and identify their actual causes.

Also, one problem may be related to other problems and solving one problem without having regard to the others may make the matters worse. It is essential, therefore, that an analysis be made as to how the solution to one problem is likely to affect other problems or the situation in general. Further, it is likely for an organisation to have several problems. However, an analyst usually cannot deal with all the problems and has to focus only on a few of them. For most companies it translates in to selecting those problems whose solutions are likely to result in greatest profit increases or cost reductions. Thus, it is imperative that right problems be selected for solution. In sum, it is necessary for an analyst to understand that the formulation of a problem develops from a complicated interaction that involves the selection and interpretation of data between him and the management.

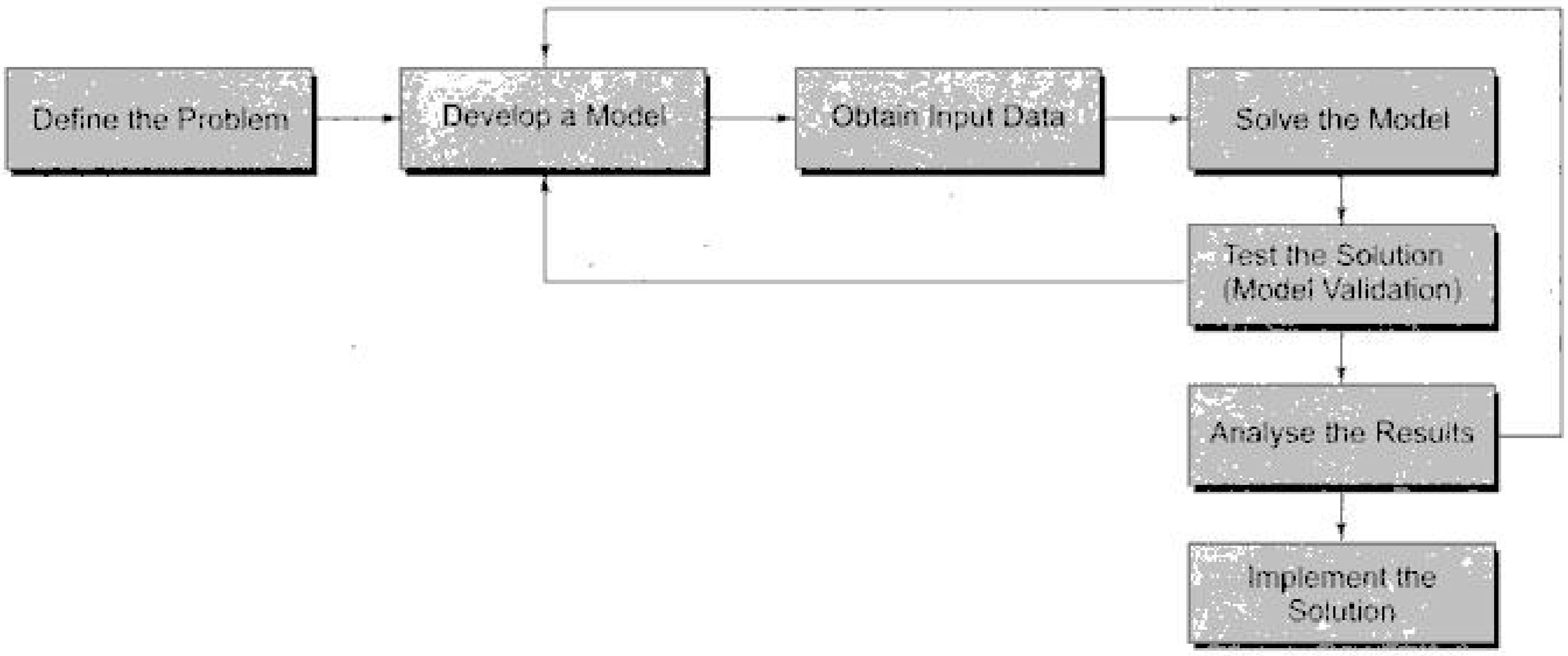


Fig. 1.1 Operations Research Methodology

Once the problem has been identified, it is categorised as being standard or special. The *standard problems* are also known as *programmed problems.* As has already been mentioned, they are the well-structured problems characterised by routine, repetitive decisions that utilise specific decision-making techniques in their solution strategy. Standard solution procedures have been developed to handle such prototype problems. Examples of these problems include the assignment of workers to jobs, fixing the product-mix for a month and determination of the quantity of materials to be bought. On the other hand, there are *special* or *non-programmed* problems. They are unique and non-recurrent in nature and, therefore, ill-structured. Undertaking of a research and development project and the merger and consolidation decisions illustrate such type of decision situations.

2. Model Building Once the problem is defined, the next step is to build a suitable model. As already mentioned, the concepts of models and model-building lie at the very heart of the operations research approach to problem solving. A model is a theoretical abstraction of a real-life problem. In fact, many real-life situations tend to be very complex because there are literally innumerable inherent factors in any given situation. Thus, the decision-maker has to abstract from the empirical situation those factors which are most relevant to the problem. Having selected the critical factors, he combines them in some logical manner so that they form a counterpart or a model of the actual problem.

Thus, a model is a simplified representation of a real-world situation that, ideally, strips a natural phenomenon of its bewildering complexity and replicates its essential behaviour. Models may be represented in a variety of ways. They can be classified as physical and symbolic models.

(a) Physical Models

A physical model is a physical or schematic representation of the real thing. There are two types of physical models: iconic and analogue.

(i) Iconic Models: They are essentially the scaled-up/down versions of the particular thing they represent. A model aeroplane in a wind tunnel, a model of a proposed building provided by an architect, models of the sun and its planets housed in a planetarium, a model of a particular molecular structure of a chemical are examples of iconic models, because they look like what they represent (except size). Maps, pictures or drawings may also be categorised as iconic models since they represent essentially the images of certain things.

The chief merit of an iconic model is that it is concrete and specific. It resembles visually the thing it represents and, therefore, there are likely to be fewer problems in translating any 'findings' from the model into the real-life situation. However, the disadvantage of such models is that they often do not lend themselves to manipulation for experimental purposes.

(ii) Analogue Models The analogue models use one set of properties to represent another set. To illustrate, an electrical network model may be used as an analogue model to study the flows in a transportation system. Similarly, a barometer that indicates changes in atmospheric pressure through movements of a needle is an example of analogue model and the contour lines on a map are analogues of elevation. In general, the analogue models are less specific and concrete but they are easier to manipulate as compared to the iconic models.

(b) Symbolic Models

Many real-life problems can be described by symbolic models or mathematical forms. These are the most general and abstract types of models. They employ letters, numbers and other types of symbols to represent the variables and their interrelationships. As such, they are capable of experimental manipulation most easily. The symbolic models can be verbal or mathematical. Whereas the verbal models describe a situation in spoken language or written words, the mathematical models employ mathematical notation to represent, in a precise manner, the variables of the real situations. The mathematical models take the form of mathematical relationships that portray the structure of what they are intended to represent. The use of a verbal versus mathematical model could be shown by the formula for finding the perimeter of a rectangle. A verbal model would express this problem as follows. The perimeter (P) of a rectangle is equal to the sum total of two times the length (L) and two times the width (W) of the rectangle. In contrast, the advantage of the mathematical model is demonstrated by the following statement: $P = 2L + 2W$. If applied to the same rectangle, both models would yield identical results. However, a mathematical model is more precise.

Symbolic models are used in operations research because they are easier to manipulate and they yield more accurate results under manipulation compared to the iconic or analogue models.

Use of Mathematical Models Various types of mathematical models are used in modern operations research. Two broad categories of these are deterministic and probabilistic models. A *deterministic model* is the one in which all parameters in the mathematical formulation are fixed at predetermined values so that no uncertainty exists. In a *probabilistic model*, on the other hand, some or all the basic characteristics may be random variables (capable of assuming different values with given probabilities). In such models, uncertainty and errors are required to be given explicit consideration. Probabilistic models are also termed as stochastic or chance models.

The mathematical models comprise three basic components: decision variables, result variables and uncontrollable variables. The *decision variables* represent those factors where a choice could be made. These variables can be manipulated and, therefore, are controllable by the decision-maker. The *result variables* indicate the level of effectiveness of a system. They represent output of the system and are also termed as *dependent variables* but are beyond the control of the decision-maker. To illustrate, in the area of marketing, the decision variables may be the advertising budget, the number of regional salesmen employed, the number of products and so on; result variables may be the market share for the company, level of customer satisfaction and others; while the uncontrollable variables may be the competitors' strategies, consumer incomes and so on.

As mentioned earlier, the different components of a mathematical model are tied together with the relationships in the form of equations, inequalities and so on. Such a model consists of an objective function and describes how a dependent (result) variable is related to independent (decision) variables. For example, the profit function of a firm making two products can be stated as follows:

$$P = p_1x_1 + p_2x_2$$

in which P indicates the total profit of the firm, x_1 and x_2 are the number of units (independent) variables of the two products produced and sold, and p_1 and p_2 the profit per unit on the two products, respectively (the uncontrollable variables).

The objective function is called for to maximise (or minimise), subject to certain constraints (representing the uncontrollable variables). For example, in this case of production, the firm might be able to sell no more than a certain number of units, say 80. Then the marketing constraint (an uncontrollable variable) can be expressed as follows:

$$x_1 + x_2 \leq 80$$

Similarly, other constraints, if any, of the system can be expressed.

3. Obtaining Input Data Once an appropriate model has been formulated, the next step is to obtain the data to be used in the model as input. Since the quality of data determines the quality of output, the importance of obtaining accurate and complete data cannot be over-emphasised. Obviously, the finished product can be no better than the raw materials used. This situation may be described as GIGO: gold in, gold out or garbage in, garbage out.

Obtaining correct and relevant data may indeed be a difficult exercise when relatively large problems are involved. A number of sources, including company reports and documents, interviews with the company personnel and so on, may be used for collecting data.

4. Solution of Model Having formulated an appropriate model and collected the input data, the next stage in the analysis calls for the solution to the model and the interpretation of the solution in context of the given problem. A solution to a model implies determination of a specific set of decision variables that would yield a desired level of output. The desired level of output, in turn, is determined by the principle of choice adopted and represents the level which 'optimises'. Optimisation might mean maximising the level of goal attainment from a given set of resources or minimisation of cost as will satisfy the required level of goal attainment, or maximisation of ratio of the goal attainment to cost.

It may be noted that the solutions can be classified as being feasible or infeasible, optimal or non-optimal and unique or multiple.

(a) Feasible and Infeasible Solutions

A solution (a set of values of the decision variables, as already mentioned) which satisfies all the constraints of the problem is called a feasible solution, whereas an infeasible solution is the one which does not satisfy all the constraints. Since an infeasible solution fails to meet one or more of the system requirements, it is an unacceptable one. Only feasible solutions are of interest to the decision-maker.

(b) Optimal and non-optimal Solutions

An optimal solution is one of the feasible solutions to a problem that optimises and is, therefore, the best among them. For example, for a multi-product firm working under some given constraints of capacity, marketing, finance and so on, the optimal solution would be that product-mix which would meet all the constraints and yield the maximum contribution margin towards profits. The feasible solutions other than the optimal solution are called non-optimal solutions. To continue with the example, several other product-mixes would satisfy the restrictions imposed and hence qualify for acceptance, but they would be ignored because lower contribution margins would be associated with them. They would be non-optimal.

(c) Unique and Multiple Solutions

If only one optimal solution to a given problem exists, it is called a unique solution. On the other hand, if two or more optimal solutions to a problem exist, which are equally efficient, then multiple optimal solutions are said to exist. Of course, these are preferable from the management's point of view as they provide a greater flexibility in implementation.

Once the principle of choice has been specified, the model is solved for optimal solution. For this, the feasible solutions are considered and of them the one (or more) that optimises is chosen. For this purpose, a complete enumeration may be made so that all the possible solutions are checked and evaluated. However, this approach is limited to those situations where the number of alternatives is small. Alternatively, and more commonly, methods involving algorithms may be used to get optimal solutions. It is significant to note that in contrast to complete enumeration, where all solutions are checked, an algorithm represents a trial-and-error process, where only a part of the feasible solutions are considered and the solutions are gradually improved until an optimal solution is obtained.

While algorithms exist for most of the standardised problems, there are also some numerical techniques, which yield solutions that are not necessarily optimal. Heuristics and simulation illustrate those methods. *Heuristics* are step-by-step logical *rules,* which, in a certain number of steps, yield some acceptable solution to a given problem. They are applied in those cases where no algorithms exist. Similarly, the technique of *simulation* is also applied where a given system is sought to be replicated and experimented with. Solutions using simulation need not be optimal because the technique is only descriptive in nature. Once a solution is obtained, it needs to be tested completely before it can be analysed and implemented. Because the solution depends on the input data and the model employed, both need testing. Such testing comprises of determining the accuracy and completeness of the data used by model and ensuring that the model used is logical and adequately represents the real situation.

Sensitivity Analysis In addition to the solution to the model formulated by any technique, sensitivity analysis, or post-optimality analysis, should also be performed. By sensitivity analysis we imply determination of the behaviour of the system to changes in the system inputs and specifications. Thus, it is *what if* analysis. This is done because the input data and the structural assumptions of the model may not be valid.

5. Model Validation The validation of a model requires determining whether the model can adequately and reliably predict the behaviour of the real system that it seeks to represent. Also, it involves testing the structural assumptions of the model to ascertain their validity. Usually, the validity of a model is tested by comparing its performance with the past data available in respect of the actual system. There is, of course, no assurance that the future performance of the system will continue to be in the same manner as in its past. Therefore, one must take cognisance of the changes in the system over time and adjust the model as required.

6. Implementation The final step is the implementation of the results. It is the process of incorporating the solution in the organisation. Implementation of solution is often more difficult than it may apparently seem. No standard prescription can be given, which would ensure that the solution obtained would automatically be adopted and implemented. This is because the techniques and models used in operations research may sound such and may be detailed in mathematical terms, but they generally do not consider the human aspects that are significant in implementation of solution. The impact of a decision may cut across various segments of the organisation, and the factors like resistance to change, desire to be consulted and informed, motivation, and so on may come in the way of implementation. Equally important as the skill and expertise needed in developing a model is the requirement of tackling issues related to the factors which may have a bearing on the implementation of a solution in a given situation. Thus, a model, that secures a moderate theoretical benefit and is implemented, is better than a model which ranks very high on obtaining theoretical advantage but cannot be implemented. In fact, the importance of having managers in the organisation who would act on the results of the study of a team that analyses the problem, can hardly be over-emphasised.

1.3 QUANTITATIVE ANALYSIS AND COMPUTER-BASED INFORMATION SYSTEMS

Quantitative analysis/operations research has become an integral part of the modern computer-based information systems. A computer-based information system comprises:

Hardware:	input, CPU, storage, and output,
Software:	general operating software, general and specialised application software,
Files:	tapes, disks, documents,
Procedures:	user, input and operating procedures,
People:	managers, analysts, technical support personnel and operating personnel, and
Database:	information about various facets of the organisation.

Such systems may include management information system (MIS), decision support systems (DSS) and the use of artificial intelligence. Quantitative analysis tools are used in each of these sub-systems.

1. Management Information Systems A management information system represents an organised way of managing information and data, which are vital organisational resources that are essential to the efficient and effective operations. Thus, it comprises a body of organised procedures for identification, collection, processing, retrieval and dissemination of information. It aims at providing right information to the right people in the right place at the right time. Provision of right information can often involve help of quantitative analysis. To illustrate, if a manager needs help to take ordering and inventory stocking decisions, forecasting models for projecting demand, inventory models to determine order quantities, and so on, are going to be quite helpful. To be able to have the needed information, it is necessary that the manager is able to interact with the computers 'on-line'. In case of complexities, the quantitative analyst can aid the manager to retrieve the desired information from the system, by providing programs to the manager for the same.

2. Decision Support Systems The advances in computer technology have witnessed development of decision support systems (DSS) and artificial intelligence. In a decision support system, a system is developed to aid management in improving its decision-making. It supports, rather than replaces, managerial judgement. The presence of such a system often implies the use of computers to assist the managers in decision-making for semi-structured problems. Decision support system is interactive and allows the manager the use of *what if* questions, so that she may try different decisions, use modified data and witness results of such changes quickly. Its emphasis is on effectiveness of decision-making rather than on efficiency. Break-even analysis and profitability decisions, decision tables and expected values, decision trees, relevance trees, etc., are all typical DSS models.

3. Artificial Intelligence and Expert Systems Although machine intelligence had been dreamt of for ages, the term *artificial* intelligence, to denote demonstration of intelligence by machines, evolved during 1950s. Artificial intelligence includes the ability of computer systems and technology to think, see, learn, understand and use common sense—in other words, to mimic the human behaviour. It has grown into several important sub-systems that have broad implications for quantitative analysis. Expert systems, an offshoot of the research on artificial intelligence, are information systems that attempt to support or automate decision-making and act like intelligent and rational decision-makers. They do this by storing and using knowledge about a specific, limited topic and produce conclusions based on the data that they receive.

1.4 PLAN OF THE BOOK

As the title suggests, this book is an introduction to quantitative methods as an aid to managerial decision-making. Important quantitative techniques that may be employed by a manager in the evaluation of alternatives constitute the text of the book. The managerial problems and the tools that are used in handling them are discussed in the form of chapter scheme.

Chapters 2, 3 and 4 discuss ways and means of approaching the problems of resource allocation, under the general category of *linear programming*. The attempt here is to seek an optimal solution to allocations where (i) there are a number of activities to be performed, (ii) there exist at least two different ways to perform these activities, and (iii) resources are limited. The variables involved are assumed to be related linearly, and the boundaries or constraints of their interaction are also linear. Chapter 2 discusses the formulation of linear programming problems and the graphic approach to their solution, while Chapter 3 describes their solution using simplex method. Chapter 4 is devoted to post-optimality analysis and duality. *Transportation* and *transhipment* problems that deal with movement of a commodity from several origins to several destinations at a minimum cost are discussed in Chapter 5. While the transportation problem allows goods to be moved from various sources to different destinations, the transhipment problem permits goods being transhipped *en route* between different sources as well. The *assignment problem*, discussed in Chapter 6, is one where certain items, like people or activities, are to be assigned on a one-to-one basis to other items such as jobs or facilities, with the objective of minimising the cost or maximising the effectiveness of assignment.

Chapter 7 is addressed to *integer programming* problems, which restrict the decision variables to assume only integer values, and *goal programming* problems, which allow a manager to specify multiple goals instead of a single goal of, say, profit maximisation. In this chapter, the problem of determining the optimal way of scheduling the tour for a *travelling salesman* is also considered. Chapter 8 considers the problem of *sequencing*—the scheduling of different jobs to be performed on various machines. The next chapter deals with the *inventory* process and the question of how many units should be maintained in inventory, including how often and how many units should be ordered, and the holding of safety stocks.

Queuing theory is the subject of discussion in Chapter 10. Whenever persons or objects reach a service station, a waiting line is likely to form, particularly during the rush hours. Thus, there may be a queue at certain times while the service facility may be idle at others. In general, the larger the facility, the costlier the operation and smaller the waiting line. The problem is to determine such a service level that will minimise the relevant costs. The question of *replacement* of assets that wear out gradually and those that fail suddenly are considered in Chapter 11. Chapter 12 is addressed to the tools of *Programme Evaluation Review Technique* (PERT) and *Critical Path Method* (CPM), which are extremely useful for the purpose of planning and controlling complex projects.

Chapter 13 reviews the decision process in the structured framework of *decision theory*. Several principles of decision-making are discussed in the chapter. Also given are the posterior analysis, decision tree and utility approaches to decision-making. The next chapter contains a discussion of Markov chains, which deal basically with the problem of brand switching by the customers. It discusses how market shares are likely to be affected under conditions of customers shifting preferences from one brand to another. Chapter 15 is titled *theory of games* and deals with situations of conflict-where decisions are required to be made with the opponent being a rational decision-maker.

Dynamic programming, which is an important quantitative tool to deal with a large number of problems, is described in Chapter16. In this method, a problem is solved by dissecting it into a number of sub-problems. The solution to the sub-problems one-by-one eventually leads to the solution to the given problem.

Chapter 17 introduces a powerful tool of quantitative analysis—the *simulation*. The Monte-Carlo analysis is introduced which is a technique of testing alternatives through trial and error approaches. In effect, a system is sought to be 'copied', and through the technique of selecting random sequences of numbers from a probability distribution, we are able to test or simulate the outcomes associated with the alternatives that do not lend themselves to direct analysis and comparison.

In Chapter 18 we consider the concept of the value of money, investment analysis and the break-even analysis. An attempt is made in this chapter to see how can the risk associated with an investment proposal be quantified and appropriate capital budgeting decisions made. Finally, the *forecasting techniques* are dealt with in Chapter 19. The forecasting techniques are used for making predictions of future demand or some other variable. They are helpful to the manager in as much as the manager is required to take decisions about future which is at best uncertain.

KEY POINTS TO REMEMBER

- Decision-making is an all-pervasive feature of management.
- A systematic approach to decision-making calls for application of scientific methods.
- Decision-making may involve decisions under certainty or uncertainty, under static or dynamic conditions, and against nature or some rational opponent.
- Qualitative factors should be given due consideration along with quantitative analysis of a problem.
- Quantitative analysis/operations research provides a systematic approach to decision-making.
- Features of operations research include decision-making, scientific approach, objectivity, inter-disciplinary approach and use of digital computers.
- The methodology of operations research comprises of (i) problem formulation, (ii) model building, (iii) data collection, (iv) solution of the model and interpretation of results, (v) model validation, and (vi) implementation of the solution obtained.

- Models may be physical (including iconic and analogue) or symbolic (mathematical).
- Mathematical models have three components: decision variables, result variables and uncontrollable variables.
- Quantitative analysis is now an integral part of the modern computer-based information systems, decision support systems and artificial intelligence and expert systems.

TEST YOUR UNDERSTANDING

Mark the following statements as T *(True)* or F *(False)*

1. Decision-making is purely an art even in the modern age.
2. Decisions taken only on the basis of quantitative analysis can be sound and correct.
3. For a real decision-making situation, definition and identification of alternatives, the states and consequences are the most difficult aspects.
4. An orderly and organised way of thinking is needed in order to formulate a problem in to a well-structured framework for solution.
5. Quantification of a problem situation represents sound business judgement and it enables the solution to be implemented without second thoughts.
6. Operations research is inter-disciplinary in nature and requires a team approach to the solution to a problem.
7. As real-life decision situations tend to be rather complex, the decision-maker has to abstract from a given empirical situation those factors which are most relevant to the problem and combine them in some logical manner as to form a model of it.
8. Non-programmed decision-making problems are ill structured.
9. Analogue models are essentially the scaled up/down versions of the particular thing they seek to represent.
10. A feasible solution is one which satisfies all the constraints of the given problem, while an infeasible solution is one which satisfies none of them.
11. Validation of a model implies determining if the model does adequately and reliably predict the behaviour of the real system that it seeks to represent.
12. Sensitivity analysis is *what if* analysis, and deals with determining how the output changes in response to changes in the inputs.
13. Implementation of solution determined from quantitative analysis is trivial because such a solution is the outcome of rigorous mathematical analysis.
14. A management information system (MIS) comprises a body of organised procedures for identification, collection, processing, retrieval and dissemination of information, aiming at providing right information to right people in right place at right time.
15. Decision support systems (DSS) are a good substitute for managerial judgement.

EXERCISES

1. It is said that management is equivalent to decision-making. Do you agree? Explain.
2. What are the essential characteristics of operations research? Mention different phases in an operations research study. Point out some limitations of operations research. Explain the role of computers in this field.

3. Discuss the role and scope of quantitative methods for scientific decision-making in business management.
4. "Executives at all levels in business and industry come across the problems of making decisions at every stage in their day-to-day activities. Operations Research provides them with various quantitative techniques for decision-making and enhances their ability to make long range plans and solve everyday problems of running a business and industry with greater efficiency, competence and confidence." Comment with examples. *(MBA, Delhi, April, 1998)*
5. Write an essay on the scope and methodology of operations research, explaining briefly the main phases of an OR study and techniques used in solving OR problems
6. What is Operations Research? Account for the growing importance of Operations Research in business decisions. *(M Com, Delhi, 1997)*
7. Most operations research applications possess certain distinguishing characteristics. These could be identified as follows:
   (a) primary focus on decision-making;
   (b) an investigation based on some measurable criteria;
   (c) the use of a formal mathematical model; and
   (d) dependence upon computing facilities.

   Explain each of these.
8. (a) Define an OR model and give four examples. State their properties, advantages and limitations.
   (b) State the phases of an OR study and their importance in solving problems. *(CA Nov., 1986)*
9. Model building is a central element in operations research method. Give a description of the following basic types of models: (a) iconic, (b) analogue, and (c) mathematical (symbolic), with particular reference to the special insights provided by each.

   Give an account of the information requirements, assumptions and applications of any three of the following different type of mathematical models: (a) allocation, (b) queuing, (c) inventory, (d) replacement. *(ICMA, May, 1979)*
10. What is a 'model'? Distinguish between
    (a) analogue and iconic models, and
    (b) deterministic and stochastic models.
11. (a) "Statistics is the nerve center for Operations Research." Discuss.
    (b) State any four areas for the application of OR techniques in Financial Management, and how it improves the performance of the organisation. *(CA, May, 1986)*
12. All quantitative techniques have hardly any real-life applications." Do you agree with the statement? Discuss. *(MIB, Delhi, 1999)*
13. Comment on the following statements:
    (a) Operational Research is an art of winning war without actually fighting it.
    (b) Operational Research is no more than a quantitative analysis of the problem.
    (c) Operational Research is a war against *ad hocism*.
    (d) Operational Research is the art of finding bad answers where worst exist.
    (e) Operational Research advocates a systems-approach and is concerned with optimisation. It provides a quantitative analysis for decision-making.
    (f) Operational Research replaces management-by-personality.

14. It is common in business to insure against the occurrence of events, which are subject to varying degrees of uncertainty, for example, ill health of senior executives. At the same time, the use of formal analytical models to assist in the process of making decisions on business problems which are generally subject to uncertainty, does not appear to be very widespread.

    Describe the model building approach to the analysis of business problems, under conditions of uncertainty. Discuss the apparent inconsistency in company's willingness to insure when formal analytical of an Operations Research nature which allow for uncertainty are relatively rarely employed.

    (ICMA, Nov., 1981)

Chapter 2

Linear Programming I: Formulation and Graphic Solution

Chapter Overview

We live in a world of shortages since our resources which can be supplied are limited. This is all the more true at the micro level. Thus, there is always a problem as to how to allocate the given resources in the best possible manner. Linear programming is a technique which provides the answer in a wide variety of cases.

Some situations where a manager can use linear programming include the following:

- *How to allocate the advertising budget among various alternate advertising media which have different degrees of effectiveness in reaching audiences and involve different costs?*
- *In case of a multi-product firm, what product-mix will yield the maximum profit, when different products are known to have different profitability co-efficients and different resource requirements?*
- *How should the given funds be allocated between different investment opportunities that yield varying returns and involve different degrees of risk?*
- *How should a dietician decide about the foods that contain varying proportions of ingredients like carbohydrates, vitamins, proteins, etc. to be given to the patients so that their nutrition requirements are met with at the minimum cost?*
- *How should the land on an agricultural farm be allocated between different crops which involve different costs on account of labour, manure, seeds, etc. and have different yields, resulting in unequal profitability of the agricultural products produced?*
- *How should the HR manager of a hospital decide about the employment of nurses that involves lowest cost and yet meets the requirements at different times of the 24-hour day?*

The next few chapters are devoted to a detailed account of linear programming. It must be kept in mind that the most important is to develop the skill and ability to translate a given real-life situation into a linear programming format, keeping in mind its assumptions and limitations. This chapter illustrates this with examples. Later in the chapter, the graphic solution to some such problems is provided.

The chapter heavily uses the inequalities of 'less than' and 'greater than' types and equations. You should be conversant with two-dimensional graphs and their use. Plotting of equations and inequalities on a graph and the ability to determine the space on the graph over which they are satisfied, called the feasible area, holds the key to successful graphic solution to the problems.

Chapter 2

Linear Programming I: Formulation and Graphic Solution

2.1 INTRODUCTION

A large number of decision problems faced by a business manager involve allocation of resources to various activities, with the objective of increasing profits or decreasing costs, or both. When resources are in excess, no difficulty is experienced. But such cases are very rare. Practically in all situations, the managements are confronted with the problem of scarce resources. Normally, there are several activities to perform but limitations of either of the resources or their use prevent each activity from being performed to the best level. Thus, the manager has to take a decision as to how best to allocate the resources among the various activities.

The decision problem becomes complicated when a number of resources are required to be allocated and there are several activities to perform. Rule of thumb, even of an experienced manager, in all likelihood, may not provide the right answer in such cases. The decision problems can be formulated, and solved, as mathematical programming problems.

Mathematical programming involves optimisation of a certain function, called the *objective function,* subject to certain *constraints.* For example, a manager may be faced with the problem of deciding the appropriate product mix of the four products. With the profitability of the products along with their requirements of raw materials, labour etc. known, his problem can be formulated as a mathematical programming problem taking the objective function as the maximisation of profits obtainable from the mix, keeping in view the various constraints—the availability of raw materials, labour supply, market and so on. The methods of mathematical programming can be divided into three groups: linear, integer, and non-linear programming.

This chapter and the next two are devoted to linear programming, while integer programming is discussed in Chapter 7. Non-linear programming is not considered in the book.

2.2 LINEAR PROGRAMMING

The linear programming method is a technique for choosing the best alternative from a set of feasible alternatives, in situations in which the objective function as well as the constraints can be expressed as linear mathematical functions. In order to apply linear programming, certain requirements have to be met. These are discussed here.

(a) There should be an objective which should be clearly identifiable and measurable in quantitative terms. It could be, for example, maximisation of sales, of profit, minimisation of cost, and so on.

(b) The activities to be included should be distinctly identifiable and measurable in quantitative terms, for instance, the products included in a production planning problem.

(c) The resources of the system, which are to be allocated for the attainment of the goal, should also be identifiable and measurable quantitatively. They must be in limited supply. The technique would involve allocation of these resources in a manner that would trade off the returns on the investment of the resources for the attainment of the objective.

(d) The relationships representing the objective as also the resource limitation considerations, represented by the objective function and the constraint equations or inequalities, respectively, must be *linear* in nature.

(e) There should be a series of feasible alternative courses of action available to the decision-maker which are determined by the resource constraints.

When these stated conditions are satisfied in a given situation, the problem can be expressed in algebraic form, called the *Linear Programming Problem* (LPP), and then solved for optimal decision. We shall first illustrate the formulation of linear programming problems and then consider the method of their solution.

2.3 FORMULATION OF LINEAR PROGRAMMING PROBLEMS

1. The Maximisation Case Consider the following example.

Example 2.1 A firm is engaged in producing two products, *A* and *B*. Each unit of product *A* requires 2 kg of raw material and 4 labour hours for processing, whereas each unit of product *B* requires 3 kg of raw material and 3 hours of labour, of the same type. Every week, the firm has an availability of 60 kg of raw material and 96 labour hours. One unit of product *A* sold yields Rs 40 and one unit of product *B* sold gives Rs 35 as profit.

Formulate this problem as a linear programming problem to determine as to how many units of each of the products should be produced per week so that the firm can earn the maximum profit. Assume that there is no marketing constraint so that all that is produced can be sold.

The objective function The first major requirement of an LPP is that we should be able to identify the goal in terms of the objective function. This function relates mathematically the variables with which we are dealing in the problem. For our problem, the goal is the maximisation of profit, which would be obtained by producing (and selling) products *A* and *B*. If we let x_1 and x_2 represent the number of units produced per week of the products *A* and *B* respectively, the total profit, Z, would be equal to $40x_1 + 35x_2$, because the unit profit on the two products is, respectively, Rs 40 and Rs 35. Now $Z = 40x_1 + 35x_2$ is, then, the objective function, relating the profit and the output level of each of the two items. Notice that the function is a linear one. Further, since the problem calls for a decision about the optimal values of x_1 and x_2, these are known as the *decision variables.*

The constraints As has been laid, another requirement of linear programming is that the resources must be in limited supply. The mathematical relationship which is used to explain this limitation is inequality. The limitation itself is known as a *constraint.*

Each unit of product *A* requires 2 kg of raw material while each unit of product *B* needs 3 kg. The total consumption would be $2x_1 + 3x_2$, which cannot exceed the total availability of 60 kg every week. We can express this constraint as $2x_1 + 3x_2 \leq 60$. Similarly, it is given that a unit of *A* requires 4 labour hours for its production and one unit of *B* requires 3 hours. With an availability of 96 hours a week, we have $4x_1 + 3x_2 \leq 96$ as the labour hours constraint.

It is important to note that for each of the constraints, inequality* rather than equation has been used. This is because the profit maximising output might not use all the resources to the full—leaving some unused. Hence the ≤ sign. However, it may be noticed that all the constraints, like objective function, are also linear in nature.

Non-negativity condition Quite obviously, x_1 and x_2, being the number of units produced, cannot have negative values. Thus, both of them can assume values only greater-than-or-equal-to zero. This is the non-negativity condition, expressed symbolically as $x_1 \geq 0$ and $x_2 \geq 0$.

Now, we can write the problem in complete form as follows.

Maximise	$Z = 40x_1 + 35x_2$	Profit
Subject to		
	$2x_1 + 3x_2 \leq 60$	Raw material constraint
	$4x_1 + 3x_2 \leq 96$	Labour hours constraint
	$x_1, x_2 \geq 0$	Non-negativity restriction

2. The Minimisation Case Consider the following example.

Example 2.2 The Agricultural Research Institute suggested to a farmer to spread out at least 4800 kg of a special phosphate fertiliser and not less than 7200 kg of a special nitrogen fertiliser to raise productivity of crops in his fields. There are two sources for obtaining these—mixtures *A* and *B*. Both of these are available in bags weighing 100 kg each and they cost Rs 40 and Rs 24 respectively. Mixture *A* contains phosphate and nitrogen equivalent of 20 kg and 80 kg respectively, while mixture *B* contains these ingredients equivalent of 50 kg each.

Write this as a linear programming problem and determine how many bags of each type the farmer should buy in order to obtain the required fertiliser at minimum cost.

The objective function In the given problem, such a combination of mixtures *A* and *B* is sought to be purchased as would minimise the total cost. If x_1 and x_2 are taken to represent the number of bags of mixtures *A* and *B* respectively, the objective function can be expressed as follows:

Minimise	$Z = 40x_1 + 24x_2$	Cost

The constraints In this problem, there are two constraints, namely, a minimum of 4,800 kg of phosphate and 7,200 kg of nitrogen ingredients are required. It is known that each bag of mixture *A* contains 20 kg and each bag of mixture *B* contains 50 kg of phosphate. The phosphate requirement can be expressed as $20x_1 + 50x_2 \geq 4{,}800$. Similarly, with the given information on the contents, the nitrogen requirement would be written as $80x_1 + 50x_2 \geq 7{,}200$.

* The inequalities used are not strict ones. A strict inequality is expressed as, for example, $a < b$ (or $a > b$). Here the inequalities are loose—thus permitting *a* to be smaller (or greater) than, or equal to, *b*.

Non-negativity condition As before, it lays that the decision variables, representing the number of bags of mixtures *A* and *B*, would be non-negative. Thus, $x_1 \geq 0$ and $x_2 \geq 0$.

The linear programming problem can now be expressed as follows:

Minimise	$Z = 40x_1 + 24x_2$	Cost
Subject to		
	$20x_1 + 50x_2 \geq 4800$	Phosphate requirement
	$80x_1 + 50x_2 \geq 7200$	Nitrogen requirement
	$x_1, x_2 \geq 0$	Non-negativity restriction

2.4 GENERAL STATEMENT OF LINEAR PROGRAMMING PROBLEMS

In general terms, a linear programming problem can be written as

Maximise $\quad Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad$ Objective Function

Subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \end{aligned}\right\} \text{Constraints}$$

$x_1, x_2, \ldots x_n \geq 0 \quad$ Non-negativity Restriction

where c_j, a_{ij}, b_i ($i = 1, 2 \ldots m$; $j = 1, 2, \ldots, n$) are known as *constants* and x_j's are *decision variables*. The c_j's are termed as the *profit coefficients*, a_{ij}'s the *technological coefficients* and b_i's the *resource values*. In shorter term, the problem can be written as:

Maximise
$$Z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq bi \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Where the objective is to minimise a function, the problem is,

Minimise
$$Z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq bi \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

In matrix notation, an LPP can be expressed as follows:

Maximisation Problem		*Minimisation Problem*	
Maximise	$Z = cx$	Minimise	$Z = cx$
Subject to		Subject to	
	$ax \le b$		$ax \ge b$
	$x \ge 0$		$x \ge 0$

where

c = row matrix containing the coefficients in the objective function,

x = column matrix containing decision variables,

a = matrix containing the coefficients in the constraints,

b = column matrix containing the RHS values of the constraints.

Notes

1. Generally, the constraints in the maximisation problems are of the ≤ type, and of the ≥ type in the minimisation problems. But a given problem may contain a mix of the constraints, involving the signs ≤, ≥ and/or =.
2. Usually, the decision variables are non-negative. However, they need not always be so. To illustrate, in an investment problem, if we let x_1 represent the amount to be invested in the shares of a particular company, then variable x_1 shall be non-negative since we may decide to invest ($x_1 > 0$) or not to invest ($x_1 = 0$). But, if we already hold such shares which may be sold, if the need be, then x_1 may take positive value (more investment), zero value (indicating no new investment in it) or negative value (implying disinvestment in this share). Hence, x_1 shall be ***unrestricted in sign*** or a ***free variable.***

Formulation of some typical linear programming problems is given later in this chapter under Review Illustrations.

2.5 ASSUMPTIONS UNDERLYING LINEAR PROGRAMMING

A linear programming model is based on the assumptions of proportionality, additivity, continuity, certainty, and finite choices. These are explained here.

1. Proportionality A basic assumption of linear programming is that proportionality exists in the objective function and the constraint inequalities. For example, if one unit of a product is assumed to contribute Rs 10 toward profit, then the total contribution would be equal to $10x_1$, where x_1 is the number of units of the product. For 4 units, it would equal Rs 40 and for 8 units it would be Rs 80, thus if the output (and sales) is doubled, the profit would also be doubled. Similarly, if one unit takes 2 hours of labour of a certain type, 10 units would require 20 hours, 20 units would require 40 hours . . . . and so on. In effect, then, proportionality means that there are constant returns to scale—and there are no economies of scale.

2. Additivity Another assumption underlying the linear programming model is that in the objective function and constraint inequalities both, the total of all the activities is given by the sum total of each activity conducted separately. Thus, the total profit in the objective function is determined by the sum of the profit contributed by each of the products separately. Similarly, the total amount of a resource used is equal to the sum of the resource values used by various activities. This assumption implies that there is no interaction among the decision variables (interaction is possible when, for example, some product is a by-product of another one).

3. Continuity It is also an assumption of a linear programming model that the decision variables are continuous. As a consequence, combinations of output with fractional values, in the context of production problems, are prossible and obtained frequently. For example, the best solution to a problem might be to produce $5\frac{2}{3}$ units of product *A* and $10\frac{1}{3}$ units of product *B* per week.

Although in many situations we can have only integer values, but we can deal with the fractional values, when they appear, in the following ways. *Firstly,* when the decision is a one-shot decision, that is to say, it is not repetitive in nature and has to be taken only once, we may round the fractional values to the nearest integer values. However, when we do so, we should evaluate the revised solution to determine whether the solution represented by the rounded values is a feasible solution and also whether the solution is the *best* integer solution. *Secondly,* if the problem relates to a continuum of time and it is designed to determine optimal solution for a given time period only, then the fractional values may not be rounded. For instance, in the context of a production problem, a solution like the one given earlier to make $5\frac{2}{3}$ units of *A* and $10\frac{1}{3}$ units of *B* per week, can be adopted without any difficulty. The fractional amount of production would be taken to be the work-in-progress and become a portion of the production of the following week. In this case, an output of 17 units of *A* and 31 units of *B* over a three-week period would imply $5\frac{2}{3}$ units of *A* and $10\frac{1}{3}$ units of *B* per week. *Lastly,* if we must insist on obtaining only integer values of the decision variables, we may restate the problem as an integer programming problem, forcing the solutions to be in integers only.

4. Certainty A further assumption underlying a linear programming model is that the various parameters, namely, the objective function coefficients, the coefficients of the inequality/equality constraints and the constraint (resource) values are known with certainty. Thus, the profit per unit of the product, requirements of materials and labour per unit, availability of materials and labour, etc. are given and known in a problem involving these. The linear programming is obviously *deterministic* in nature.

5. Finite Choices A linear programming model also assumes that a limited number of choices are available to a decision-maker and the decision variables do not assume negative values. Thus, only non-negative levels of activity are considered feasible. This assumption is indeed a realistic one. For instance, in the production problems, the output cannot obviously be negative, because a negative production implies that we should be able to reverse the production process and convert the finished output back into the raw materials!

2.6 SOLUTION TO LINEAR PROGRAMMING PROBLEMS—GRAPHIC METHOD

Now we shall consider the solution to the linear programming problems. They can be solved by the graphic method or by applying the algebraic method, called the *Simplex Method.* The graphic approach is restricted in application—it can be used only when two variables are involved. Nevertheless, it provides an intuitive grasp of the concepts that are used in the simplex technique.

We shall discuss the graphic method here and the use of simplex algorithm shall be explained in the next chapter.

The Graphic Method To use the graphic method for solving linear programming problems, the following steps are required:

(a) Identify the problem—the decision variables, the objective function, and the constraint restrictions.
(b) Draw a graph that includes all the constraints/restrictions and identify the feasible region.
(c) Obtain the point on the feasible region that optimises the objective function—the optimal solution.
(d) Interpret the results.

We shall demonstrate the graphical approach first in respect of the maximisation and then for the minimisation problems.

1. The Maximisation Case

We shall consider Example 2.1 again. For this problem, the decision variables are x_1 and x_2, the number of units of the products *A* and *B* respectively. The objective function and the constraints are reproduced as,

Maximise	$Z = 40x_1 + 35x_2$	Profit
Subject to		
	$2x_1 + 3x_2 \leq 60$	Raw material constraint
	$4x_1 + 3x_2 \leq 96$	Labour hours constraint
	$x_1, x_2 \geq 0$	

Graphing the restrictions We shall first chart the given restrictions on the graph. The constraint of raw material availability, $2x_1 + 3x_2 \leq 60$, states that any combination of $2x_1 + 3x_2$ that does not exceed 60 is acceptable. With varying values of x_1 and x_2, the combination can assume a maximum value equal to 60. This constraint can be depicted graphically by plotting the straight line $2x_1 + 3x_2 = 60$. Since only two points are needed to obtain a straight line, we can get the two points by setting one of these variables in turn equal to zero, and calculating the value of the other. Thus, when $x_1 = 0$, x_2 would equal 60/3 = 20, and when x_2 is set equal to 0, x_1 would be equal to 60/2 = 30. Joining the points (0, 20) and (30, 0) we get the required straight line which is shown in Figure 2.1 as line PT. Any point (representing a combination of x_1 and x_2) that falls on this line or in the area below it, is acceptable insofar as this constraint is concerned. Since it is laid that both, x_1 and x_2, are non-negative, the triangle OPT formed by two axes and the line representing the equation $2x_1 + 3x_2 = 60$ is the region containing acceptable values of x_1 and x_2 in respect of this constraint.

Similarly, the other constraint $4x_1 + 3x_2 \leq 96$ can be plotted. The line JR in Figure 2.1 represents the equation $4x_1 + 3x_2 = 96$. The triangle formed by OJR, formed by the two axes and this line represents the area in which any point would satisfy this constraint of labour hours.

Now, since both the constraints are to be satisfied, we have to consider the area on the graph that is bound by both of the constraints. A point like *C* on the graph would be unacceptable because only first of the constraints would be satisfied and not the other. Similarly, a point like *D* would also not be acceptable as it represents a combination of x_1 and x_2 which satisfies the second, and not the first, of the constraints. In fact, all points lying in the shaded area are the points of interest. They represent the combinations of x_1 and x_2 that would satisfy the given constraints. This region of acceptable values of the decision variables in relation to the given constraints (and the non-negativity restrictions) is called the *feasibility region.* In general terms, then, all inequalities of a linear programming problem, when plotted on the graph, define a space such that all points that lie within, or on the boundaries of that space simultaneously satisfy all the constraints, and the space is known as the *feasible region.* Thus, each point within this region would yield values of x_1 and x_2 as will satisfy *all* the constraints given in the problem. As such, corresponding to each point in the feasible region, we get a *feasible solution.* The feasible region of an LPP, therefore, gives a set of feasible solutions. Any point other than the one from this region will yield an *infeasible solution*—which would fail to satisfy one or more of the constraints.

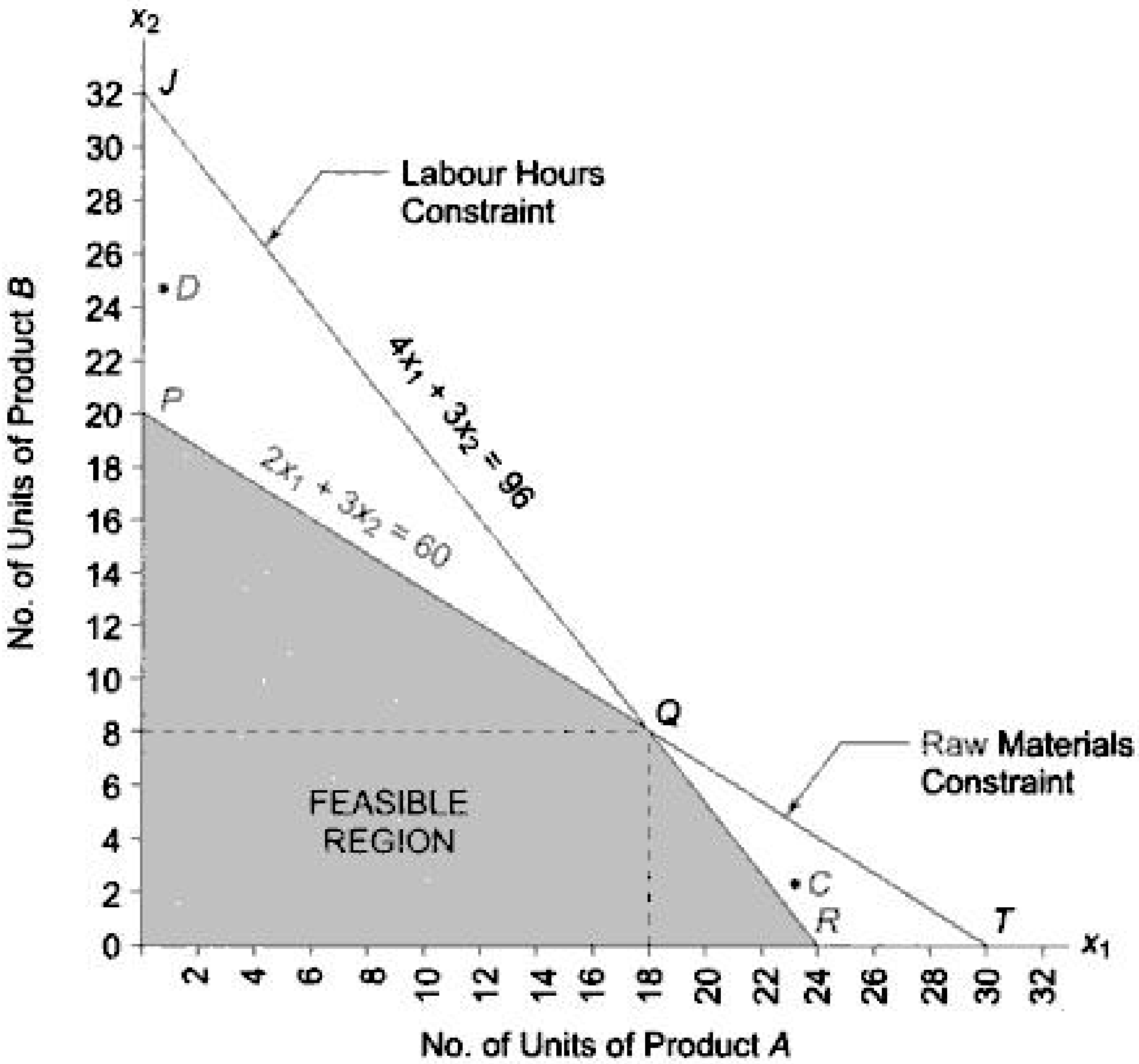


Fig. 2.1 Graphic Plot of Constraints

Obtaining the optimal solution Now we shall see as to how the optimal solution to the problem can be obtained. Although all points in the feasible region represent feasible decision alternatives, they are not all equally attractive. Some provide a greater profit contribution than others. We have to select the best point from among the infinite number of points in the feasible region. In other words, we seek to obtain the feasible solution which optimises. It requires adding one more line to the graph, called an *iso-profit* or *constant-profit* line.

As its name implies, all the points on an iso-profit line yield the same profit. Suppose, for example, that we want to find an iso-profit line representing a profit of Rs 280. For this, we put $40x_1 + 35x_2 = 280$, and plot the line on the graph. We have $x_1 = 7$ when $x_2 = 0$, and $x_2 = 8$ when $x_1 = 0$. Joining the points (7, 0) and (0, 8), we get the iso-profit line *EF* in Figure 2.2., which is a reproduction of Figure 2.1. All points on this line, representing the various combinations of the variables x_1 and x_2 shall yield a profit of Rs 280.

It may be observed that all the points that fall on this line (in the first quadrant of course) lie in the feasible region. So it is clearly possible for the firm to realise a profit of Rs 280. In fact, even greater profit can be realised by moving to other iso-profit lines, corresponding to higher profit values. Look at the Rs 840 iso-profit line, *GH*. Some of the points on this line fall outside the feasible region and, therefore, do not provide legitimate alternatives. However, other parts of this line fall in the feasible region. Therefore, a profit of Rs 840 is attainable. The iso-profit line, *JK* corresponding to the profit of Rs 1,120 lies beyond the feasible region, indicating that this profit cannot be attained.

Notice that the iso-profit lines are all parallel to each other and the farther are the lines removed from the origin of the graph, the greater is their contribution. Since all the lines have the same slope, the final step in the analysis is to continue constructing iso-profit lines that are successively farther away from the origin. This process would stop when a movement away from the origin would cause the line to lie beyond the feasible region. For example, the line that would yield the maximum profit is *LM* that passes through the point *Q*. This point gives a maximum contribution and thus represents the optimal solution. This decision is for the production of 18 units of product *A* and 8 units of product *B*, for a total profit of Rs 1000.

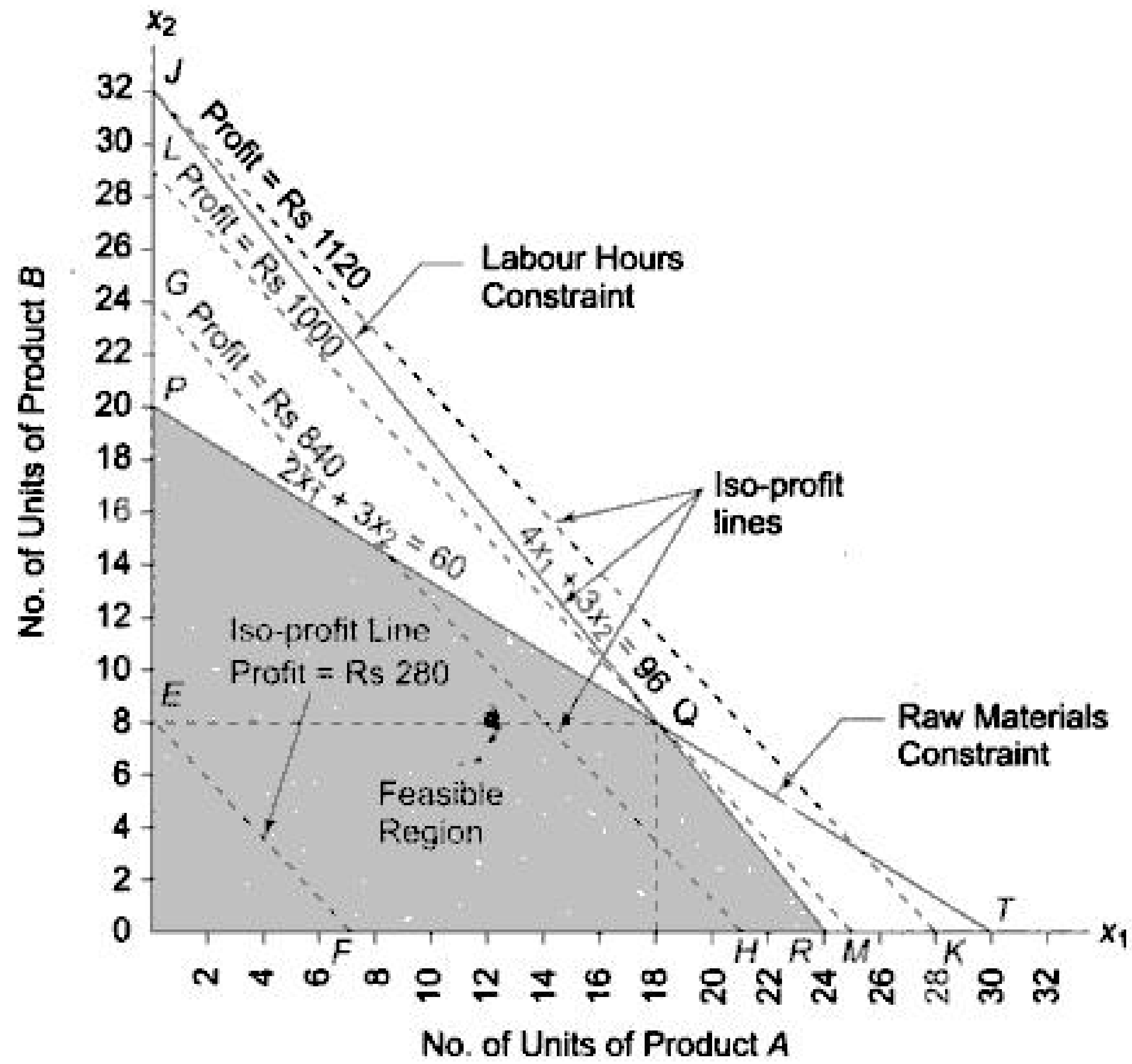


Fig. 2.2 Determination of Optimal Solution Using Iso-profit Line

The production of 18 units of *A* and 8 units of *B* will consume $2 \times 18 + 3 \times 8 = 60$ kg of raw material and $4 \times 18 + 3 \times 8 = 96$ labour hours. Hence, both the resources shall be fully utilised by this production plan.

Convex Sets and LPPs Observe the feasible region from Figure 2.1 again. It is seen to be formed by a four-sided polygon, represented by *OPQR*. Besides the two sides provided by non-negativity restrictions, each of the other two sides of the polygon is provided by a constraint. It may be observed that the feasible region determined by the constraints of the given system is a *Convex Set*.

The concept of convex set in the context of a two-variable problem can be understood as follows. If *any* two points are selected in the region and the line segment formed by joining these two points lies completely in this region, including on its boundary, then this represents a convex set. Thus, for the feasible region to be convex, no part of any line obtainable by joining a pair of points in that region should lie outside it. Figure 2.3 illustrates the difference between a convex and a non-convex set.

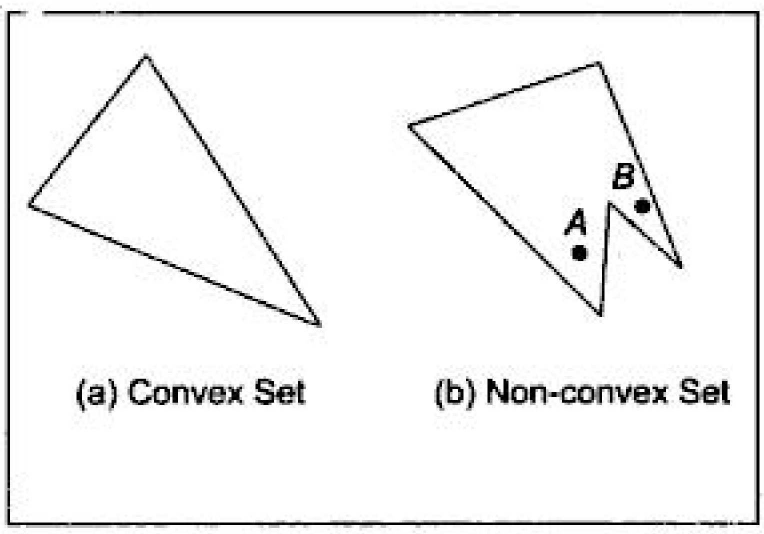


Fig. 2.3 Convex and Non-convex Sets

The set shown in part (a) of the figure is convex but the one shown in part (b) is not. For the set shown in part (b), there are many points, like *A* and *B*, for which the connecting line segment contains points that are not a part of the set as they lie outside it.

In our study of linear programming, a certain type of point in a convex set, called an *extreme point*, is of particular interest. For any convex set *S*, a point *A* in *S* is called an extreme point if each line segment that lies completely in *S* and contains point *A*, has *A* as an end point of the line segment. And, if *S* happens to be a polygon, the extreme points of *S* would be the corners or vertices of the polygon. It may be noted that the feasible region for any LPP is a convex set with only a finite number of extreme points and that an LPP that has an optimal solution has an extreme point that is optimal. This is an important observation because it reduces the set of points yielding an optimal solution from the entire feasible region containing an infinite number of points to the set of extreme points which are few in number.

Thus, our task is simplified as the optimal solution to an LPP shall be given by some corner or extreme point of the feasible region. Accordingly, in lieu of drawing an iso-profit line, we may proceed to evaluate the extreme points only to obtain the optimal solution.

In our example, the feasible region is given by the polygon *OPQR*, so we shall obtain the ordinates of each of these points (giving the values of x_1 and x_2) and calculate the objective function value by substituting them into the objective function. The values of x_1 and x_2 at each point can be read directly from the graph. Also, in respect of those points which lie on the intersection of two (or more) lines, we can get the values by solving simultaneously the equations representing those lines. The ordinates at point *Q*, for example, can be determined by solving the following pair of equations simultaneously:

$$2x_1 + 3x_2 = 60$$

$$4x_1 + 3x_2 = 96$$

This gives $x_1 = 18$ and $x_2 = 8$.

The Z-values corresponding to the various points in respect of the given problem are shown here.

Point	x_1	x_2	$Z = 40x_1 + 35x_2$	
O	0	0	0	
P	0	20	700	
Q	18	8	1000	←Maximum
R	24	0	960	

Since *Z* is highest at point *Q*, the optimal solution is to produce 18 units of product *A* and 8 units of product *B* every week, to get the profit of Rs 1000. No other product mix under the given conditions could yield more profit than this.

2. The Minimisation Case

Now we shall consider the graphical solution to the linear programming problems of the minimisation nature. Here, Example 2.2 is reconsidered.

Minimise	$Z = 40x_1 + 24x_2$	Total Cost
Subject to		
	$20x_1 + 50x_2 \geq 4{,}800$	Phosphate Requirement
	$80x_1 + 50x_2 \geq 7{,}200$	Nitrogen Requirement
	$x_1, x_2 \geq 0$	

Here the decision variables x_1 and x_2 represent, respectively, the number of bags of mixture *A* and of mixture *B*, to be bought.

Graphing the restrictions. The constraints are plotted in Figure 2.4. The first constraint of phosphate requirement, $20x_1 + 50x_2 \geq 4800$ can be represented as follows. We set $20x_1 + 50x_2 = 4800$. Putting $x_1 = 0$, we get $x_2 = 96$ and putting $x_2 = 0$, we have $x_1 = 240$. Joining the two points (0, 96) and (240, 0), we get the straight line corresponding to the above equation. The area beyond this line represents the feasible area in respect of this constraint—any point on the straight line or in the region above this line would satisfy the constraint.

Similarly, the second constraint can be depicted by plotting the straight line corresponding to the equation $80x_1 + 50x_2 = 7200$. Line *PT* in the figure represents the equation. The points falling on this line and in the area beyond it indicate the feasible values of x_1 and x_2 in respect of this constraint.

Since both the requirements are to be met, the feasible region in respect of the problem is as represented by the shaded area.

The feasible region here represents a convex set. However, it is not bounded from all the sides, as was in case of the maximisation problem. The region is unbounded on the upper side because none of the restrictions in the problem places an upper limit on the value of either of the decision variables. Obviously, if such limits are placed, the feasible region would be a bounded one.

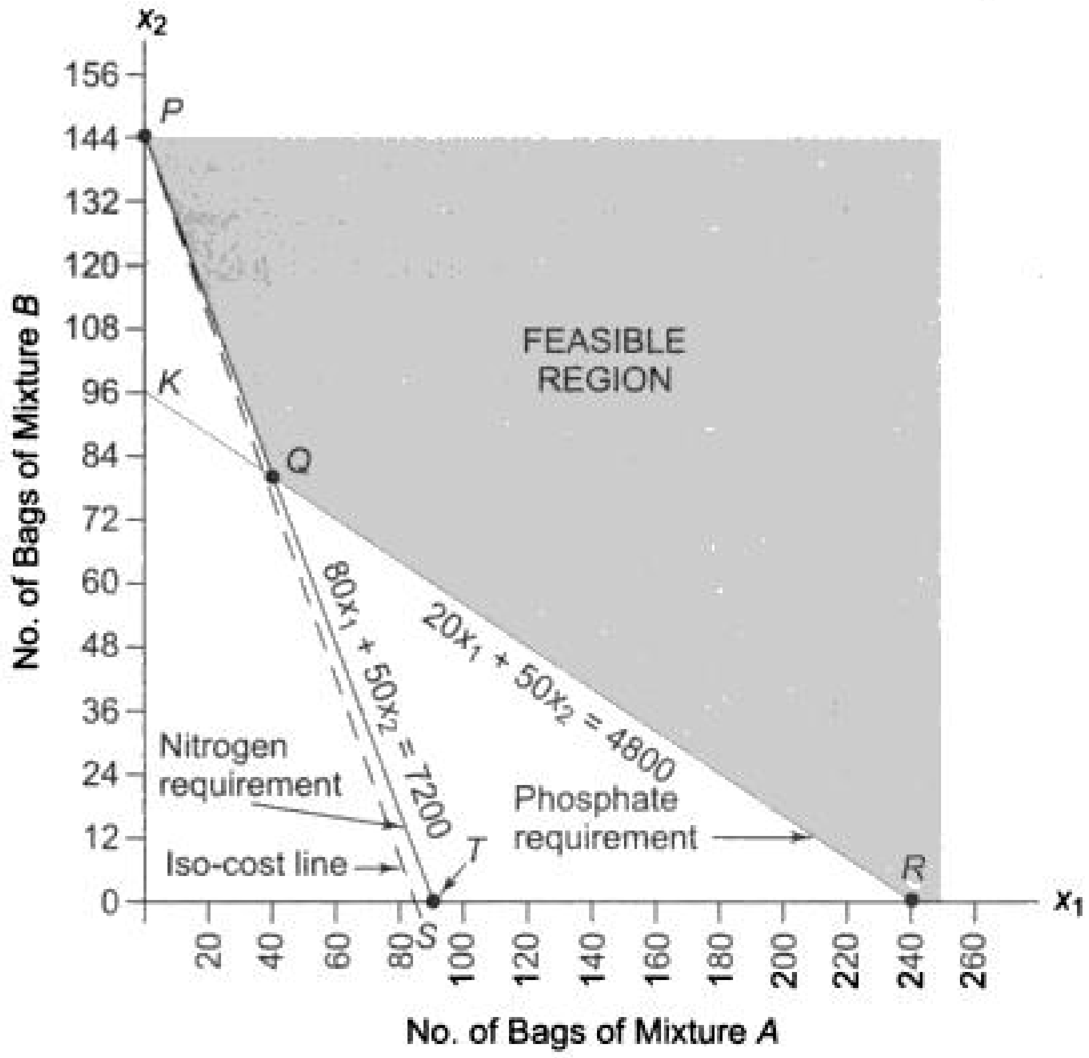


Fig. 2.4 Graphic Solution to the LPP

Obtaining the optimal solution As in case of maximisation problems, the optimal solution can also be obtained by plotting the objective function on the graph in the form of *iso-cost lines.* For drawing an iso-cost line, we assume some cost value, equate the objective function with it, and plot the straight line corresponding to that equation on the graph. Obviously, each point on an iso-cost line would imply the same cost value.

Since the total cost is to be minimised, the iso-cost line touching the feasible region, which is closest to the origin shall be the line that would provide optimal solution. In our example, the iso-cost line *PS,* which entails a cost of Rs 3,456, is such line. Since it touches point *P*, the ordinates of this point (0, 144) provide the optimal solution.

With the optimal solution of 144 bags of mixture *B* only, the supply of phosphate would be $20 \times 0 + 50 \times 144 = 7,200$ kg, which is in excess of the minimum requirement of 4,800 kg. On the other hand, the nitrogen supply would equal $80 \times 0 + 50 \times 144 = 7,200$ kg, the required minimum.

Alternately, the optimal solution to the problem may be seen to be located at an extreme point—at point *P*, *Q*, or *R* in our example. We can evaluate the ordinates at each of these points as follows:

Point	x_1	x_2	$Z = 40x_1 + 24x_2$	
P	0	144	3,456	←Minimum
Q	40	80	3,520	
R	240	0	9,600	

Since the total cost is minimum at point *P*, the optimal solution to the problem is to buy 144 bags of mixture *B* only and none of mixture *A*. This would entail a total cost of Rs 3,456.

Binding and Non-binding Constraints

Once the optimal solution to an LPP is obtained, we may classify the constraints as being *binding* or *non-binding*. A constraint is termed as binding if the left hand side and right hand side of it are equal when optimal values of the decision variables are substituted into the constraint. On the other hand, if the substitution of the decision variables does not lead to an equality between the left and the right hand sides of the constraint, it is said to be non-binding. In Example 2.1, the optimal values of decision variables are: $x_1 = 18$ and $x_2 = 8$. Substituting these values in the two constraints we get

$$2x_1 + 3x_2 \quad \text{or} \quad 2 \times 18 + 3 \times 8 = 60 = \text{RHS, and}$$

$$4x_1 + 3x_2 \quad \text{or} \quad 4 \times 18 + 3 \times 8 = 96 = \text{RHS.}$$

Thus, both the constraints are binding in nature. In Example 2.2, on the other hand, $x_1 = 0$ and $x_2 = 144$, the optimal values, may be substituted in the two constraints to get

$$20x_1 + 50x_2 \quad \text{or} \quad 20 \times 0 + 50 \times 144 = 7{,}200 \neq 4{,}800 \text{ (RHS), and}$$

$$80x_1 + 50x_2 \quad \text{or} \quad 80 \times 0 + 50 \times 144 = 7{,}200 = \text{RHS.}$$

Accordingly, the first of the constraints is non-binding and the second one is a binding one.

Redundant Constraint(s)

As we have observed, plotting of each of the constraints on the graph serves to determine the feasible region of a given problem. If and when a constraint, when plotted, does not form part of the boundary marking the feasible region of the problem, it is said to be *redundant*. The inclusion or exclusion of a redundant constraint obviously does not affect the optimal solution to the problem.

Example 2.3 Continuing with Example 2.1, suppose that each of the products are required to be packed. Every unit of product *A* requires 4 hours while every unit of product *B* needs 3.5 hours for packaging. Suppose that in the packaging department, 105 hours are available every week.

Under these conditions, what product mix would maximise the profit?

The problem can be restated as follows:

Maximise	$Z = 40x_1 + 35x_2$	Profit
Subject		
	$2x_1 + 3x_2 \leq 60$	Raw material constraint
	$4x_1 + 3x_2 \leq 96$	Labour hours constraint
	$4x_1 + 3.5x_2 \leq 105$	Packaging hours constraint
	$x_1, x_2 \geq 0$	

The constraints and the objective function (in the form of iso-profit line) are charted in Figure 2.5.

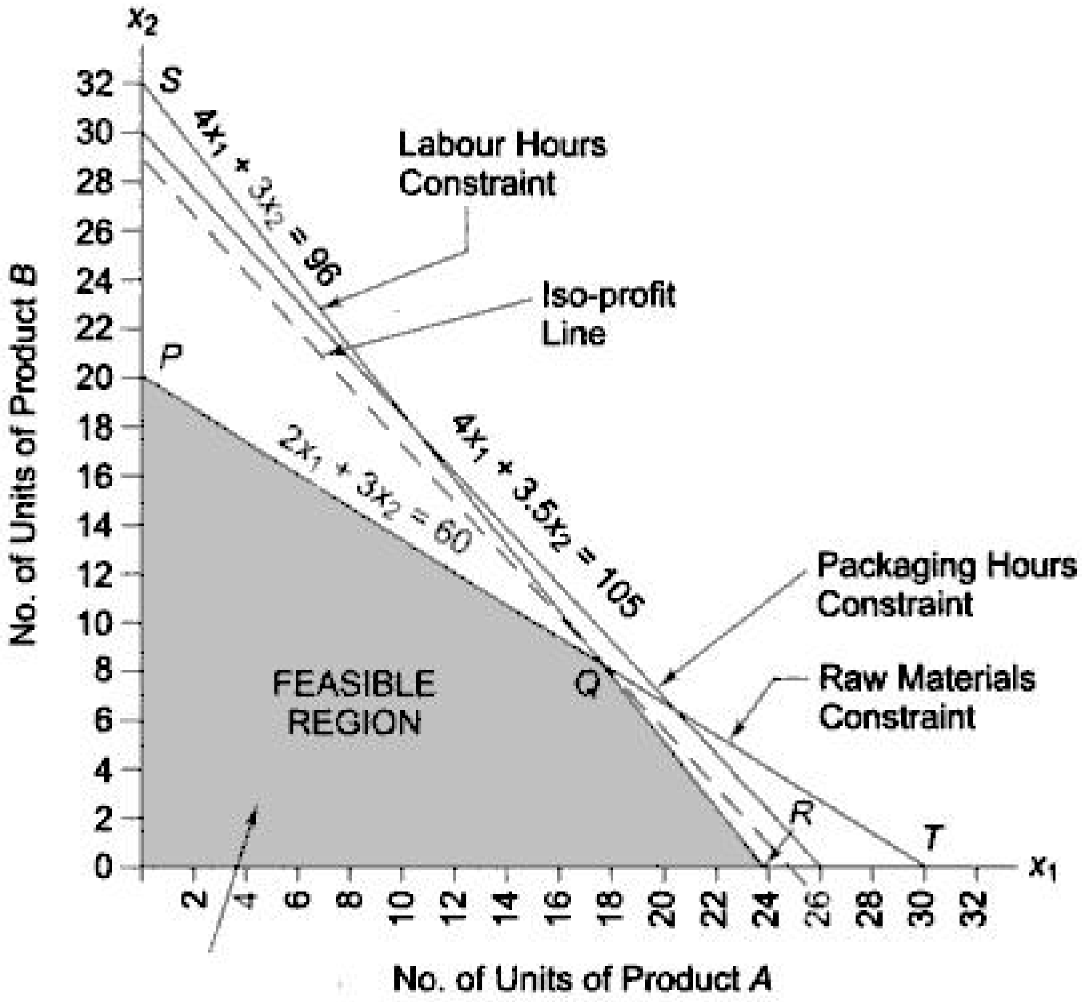


Fig. 2.5 Graphic Solution to LPP

We observe that the inclusion of the packaging hours constraint does not provide a side of the polygon representing the feasible region. Thus, this constraint is of no consequence and hence is redundant. The optimal solution to the problem is the same as the optimal solution to the problem in Example 2.1. Thus, we have $x_1 = 18$, $x_2 = 8$ representing the optimal solution. The product mix given by this shall consume the entire 60 kg of raw material and 96 labour hours, but shall leave an unused capacity of $105 - (4 \times 18 + 3.5 \times 8) = 5$ hours in the packaging department.

2.7 SOME SPECIAL CASES

In each of the three examples discussed so far, we have obtained a unique optimal solution. We now consider three types of linear programming problems which do not have unique optimal solutions. These are problems having multiple optimal solutions, having no feasible solution, and having unbounded solutions.

1. Multiple Optimal Solutions As stated above, in each of the three examples that we have considered, we have observed that the optimal solution is given by an extreme point of the feasible region and the solution is unique. The uniqueness implies that no other solution to the given problem shall yield the same value of the objective function as given by that solution. It is, of course, possible that in a given problem there may be more than one optimal solution. Each of the multiple optima would naturally yield the same objective function value.

The solution (if it exists*) to a linear programming problem shall always be unique if the slope of the objective function (represented by the iso-profit lines) is different from the slopes of the constraints. In case the objec-

* See *Infeasibility*

tive function has slope which is same as that of a constraint, then multiple optimal solutions might exist. There are two conditions that should be satisfied in order that multiple optimal solutions exist.

(a) The objective function should be parallel to a constraint that forms an edge or boundary on the feasible region; and

(b) The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint must be a binding constraint.

To fully understand, let us consider the following examples.

Example 2.4 Solve graphically the following LPP:

Maximise $Z = 8x_1 + 16x_2$

Subject to

$$x_1 + x_2 \le 200$$
$$x_2 \le 125$$
$$3x_1 + 6x_2 \le 900$$
$$x_1, x_2 \ge 0$$

The constraints are shown plotted on the graph in Figure 2.6. Also, iso-profit lines have been graphed.

We observe that iso-profit lines are parallel to the equation for third constraint $3x_1 + 6x_2 = 900$. As we move the iso-profit line farther from the origin, it coincides with the portion *BC* of the constraint line that forms the boundary of the feasible region. It implies that there are an infinite number of optimal solutions represented by all points lying on the line segment *BC*, including the extreme points represented by *B* (50, 125) and *C* (100, 100). Since the extreme points are also included in the solutions, we may disregard all other solutions and consider only these, to establish that the solution to a linear programming problem shall always lie at an extreme point of the feasible region.

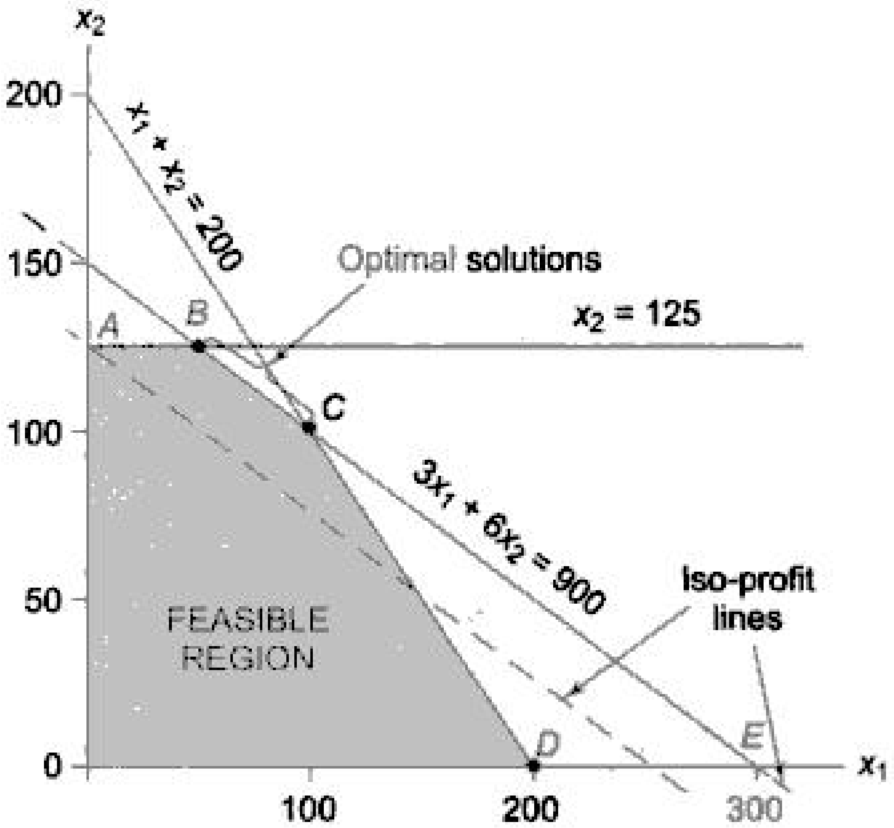


Fig. 2.6 Graphic Solution—Multiple Optima

The extreme points of the feasible region are given and evaluated here.

Point	x_1	x_2	$Z = 8x_1 + 16x_2$	
0	0	0	0	
A	0	125	2,000	
B	50	125	2,400	←Maximum
C	100	100	2,400	←Maximum
D	200	0	1,600	

The points *B* and *C* clearly represent the optima.

In this example, the constraint to which the objective function was parallel, was the one which formed a side of the boundary of the space of the feasible region. As mentioned in condition (a), if such a constraint (to which the objective function is parallel) does not form an edge or boundary of the feasible region, then multiple solutions would not exist. In Example 2.3 for instance, the objective function has the same slope as that of the packaging hours constraint (see Figure 2.5), which is a redundant constraint. Since it does not contribute to the determination of the feasible region, the problem has an optimal solution that is unique and there are no multiple solutions.

Let us consider another example.

Example 2.5 Solve graphically:

Minimise $Z = 6x_1 + 14x_2$

Subject to

$$5x_1 + 4x_2 \geq 60$$
$$3x_1 + 7x_2 \leq 84$$
$$x_1 + 2x_2 \geq 18$$
$$x_1, x_2 \geq 0$$

The restrictions in respect of the given problems are depicted graphically in Figure 2.7. The feasible area has been shown shaded. It may be observed here that although the iso-cost line is parallel to the second constraint line represented by $3x_1 + 7x_2 = 84$, and this constraint does provide a side of the area of feasible solutions, yet the problem has a unique optimal solution, given by the point *D*. Here condition (*b*) mentioned earlier, is not satisfied. This is because, being a minimisation problem, the optimal movement of the objective function would be towards the origin and the constraint forms a boundary on the opposite side. Since the constraint is not a binding one, the problem does not have multiple optima.

We can show the uniqueness of the solution by evaluating various extreme points as done here.

Point	x_1	x_2	$Z = 6x_1 + 14x_2$	
A	8	5	118	
B	84/23	240/23	168	
C	28	0	148	
D	18	0	108	←Minimum

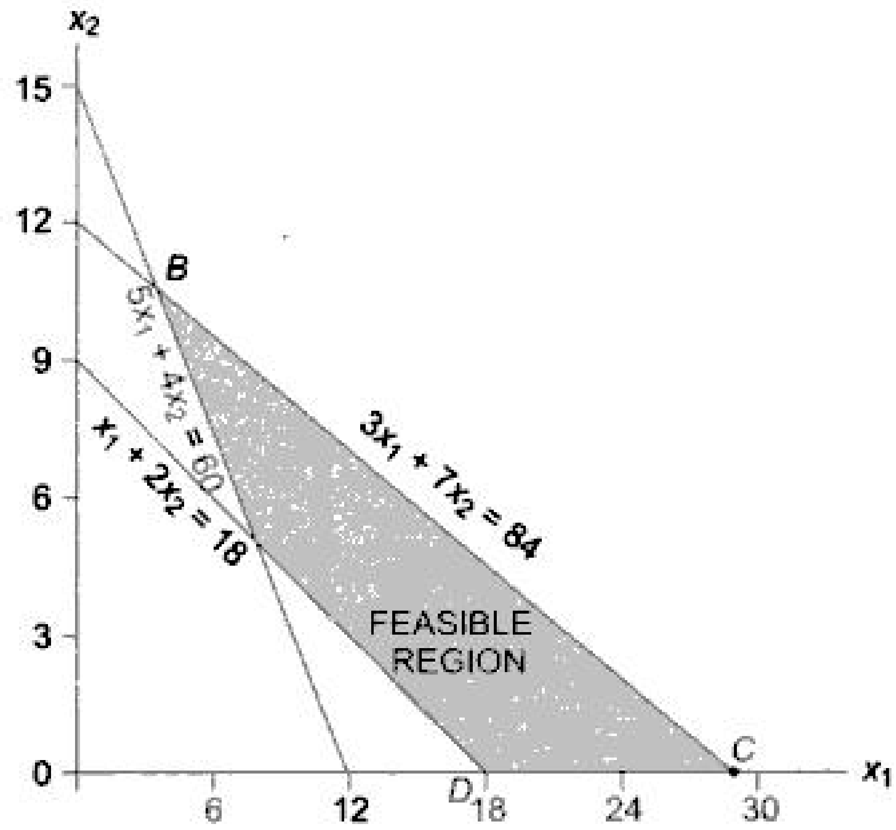


Fig. 2.7 Graphic Solution to LPP

2. Infeasibility It has already been stated that a solution is called *feasible* if it satisfies all the constraints and the non-negativity conditions. Sometimes it is possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called *infeasibility*.

In the graphic approach to the solution to an LPP, the infeasibility is evident if its feasible region is empty so that there is no feasible region in which all the constraints may be satisfied simultaneously. Consider the following example.

Example 2.6 Maximise $Z = 20x_1 + 30x_2$

Subject to

$$2x_1 + x_2 \leq 40$$
$$4x_1 - x_2 \leq 20$$
$$x_1 \geq 30$$
$$x_1, x_2 \geq 0$$

It is represented graphically in Figure 2.8. The feasible region corresponding to the first two constraints is bound by the convex set *OABC*, while the feasible region in respect of the third constraint is also shown shaded separately. We can easily observe that there is no common point in the areas shaded.

Therefore all the constraints cannot be satisfied and, as such, there is no feasible solution to the given problem.

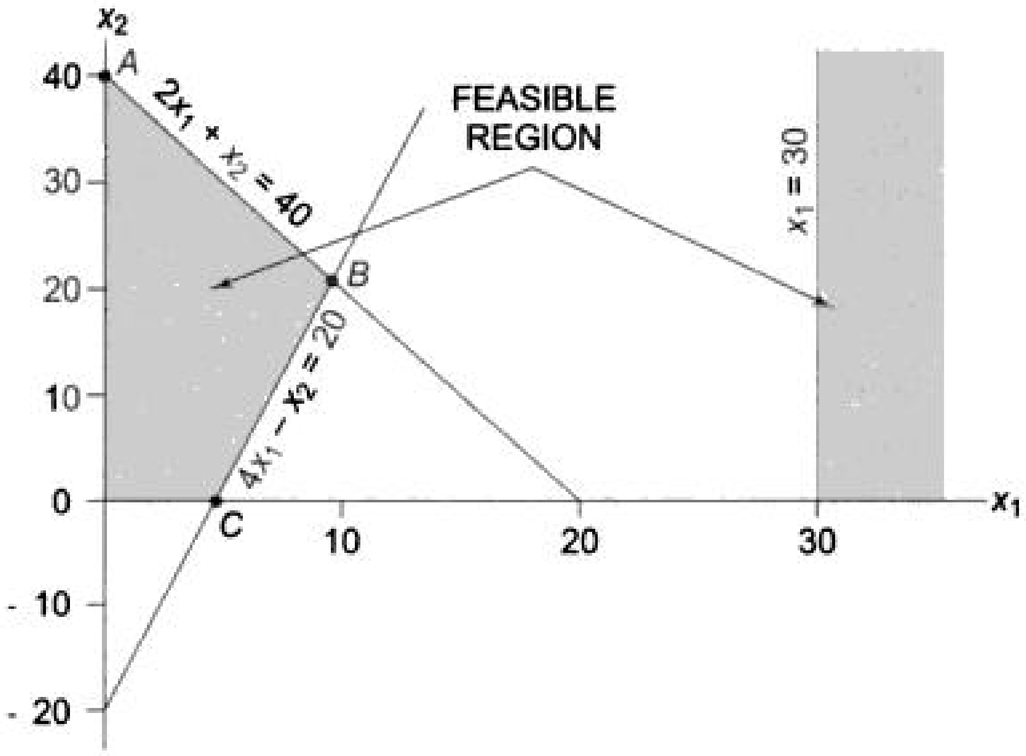


Fig. 2.8 Infeasibility

3. Unboundedness For a maximisation type of linear programming problem, unboundedness occurs when there is no constraint on the solution so that one or more of the decision variables can be increased indefinitely without violating any of the restrictions (constraints). Thus, an unbounded LPP occurs if it is possible to find points in the feasible region with arbitrarily large Z-values (may be profit or revenue). This suggests that practically if we find the solution to be unbounded for a profit-maximising linear programming problem, it may be concluded that the problem has not been correctly formulated.

Consider the following example.

Example 2.7 Maximise $Z = 10x_1 + 20x_2$

Subject to

$$2x_1 + 4x_2 \geq 16$$
$$x_1 + 5x_2 \geq 15$$
$$x_1, x_2 \geq 0$$

This is represented graphically in Figure 2.9.

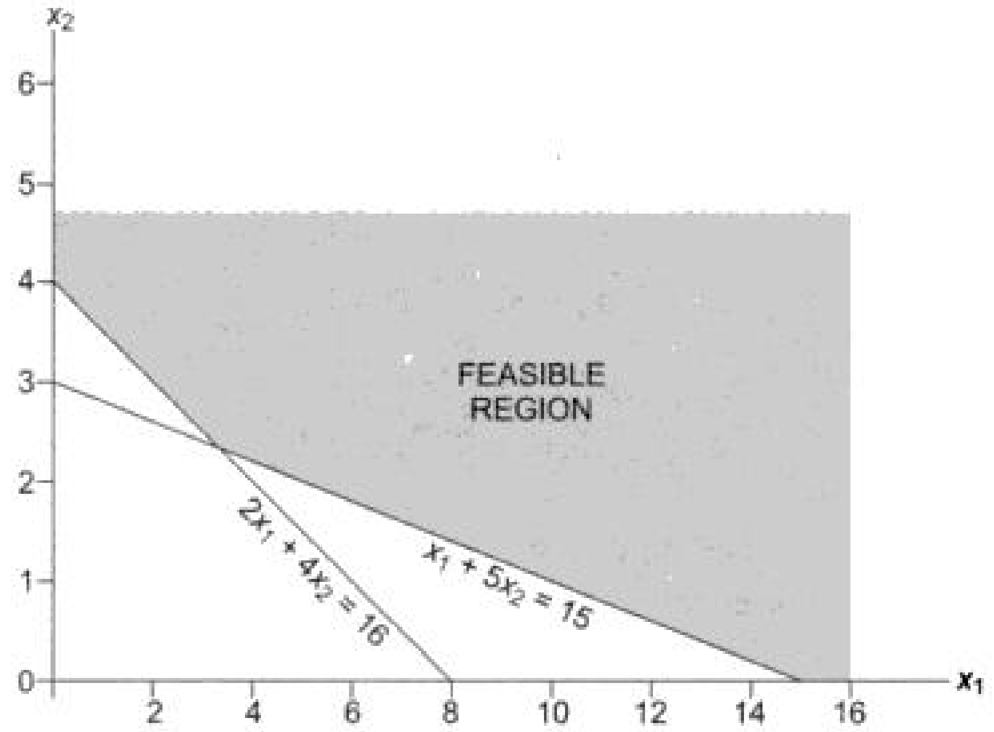


Fig. 2.9 Unboundedness

Clearly, here the objective function is not bound over the feasible region and we can move the iso-profit line upward without any limit. The problem has, therefore, unbounded solution.

For a minimisation LPP, unboundedness occurs when there are points in the feasible region with arbitrarily small values. To conclude, a maximisation LPP is unbounded if, moving parallel to the original iso-profit line in the direction of increasing Z, we never entirely leave the feasible region and, on the other hand, a minimisation problem is unbounded if we never leave the feasible region while moving in the direction of decreasing Z.

2.7.1 Infeasibility vs Unboundedness

Both infeasibility and unboundedness have a similarity in that there is no optimal solution in either case. But there is a striking difference between the two: while in infeasibility there is not a single feasible solution, in unboundedness there are infinite feasible solutions but none of them can be termed as the optimal.

Review Illustrations

Example 2.8 A 24-hour supermarket has the following minimal requirements for cashiers:

Period	:	1	2	3	4	5	6
Time of day (24-hour clock)	:	3–7	7–11	11–15	15–19	19–23	23–03
Minimum number required	:	7	20	14	20	10	5

Period 1 follows immediately after period 6. A cashier works eight consecutive hours, starting at the beginning of one of the six time periods. Determine a daily employee worksheet which satisfies the requirements with the least number of personnel. Formulate the problem as a linear programming problem.

(M.B.A., Delhi, November, 1996)

Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number of cashiers joining at the beginning of periods 1, 2, 3, 4, 5 and 6, respectively. Using the information given, the LPP may be expressed as follows:

Minimise $\quad Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to

$$
\begin{aligned}
x_1 \qquad\qquad\qquad\qquad + x_6 &\geq 7\\
x_1 + x_2 \qquad\qquad\qquad &\geq 20\\
x_2 + x_3 \qquad\qquad &\geq 14\\
x_3 + x_4 \qquad &\geq 20\\
x_4 + x_5 &\geq 10\\
x_5 + x_6 &\geq 5\\
x_i \geq 0, \quad \text{for } i = 1, 2, \ldots, 6
\end{aligned}
$$

Example 2.9 An agriculturist has a 125-acre farm. He produces radish, *muttar* and potato. Whatever he raises is sold fully in the market. He gets Rs 5 per kg for radish, Rs 4 per kg for *muttar* and Rs 5 per kg for potato. The average per acre yield is 1500 kg of radish, 1800 kg of *muttar* and 1200 kg of potato. To produce each 100 kg of radish and *muttar* and 80 kg of patato, a sum of Rs 12.50 has to be used for manure. Labour required for each acre to raise the crop is 6 man-days for radish and potato each and 5 man-days for *muttar*. A

total of 500 man-days of labour at a rate of Rs 40 per man-day is available.
Formulate this as a linear programming model to maximise the agriculturist's total profit. *(CA, May, 1997)*

Let x_1, x_2 and x_3 be the acrage for radish, *muttar* and potato, respectively. From the given information,

Output: Output of radish = $1{,}500x_1$ kg
Output of *muttar* = $1{,}800x_2$ kg
Output of potato = $1{,}200x_3$ kg

Cost of Manure: Radish : Rs 12.50 per 100 kg
Muttar : Rs 12.50 per 100 kg
Potato : Rs 12.50 per 80 kg

Accordingly,

$$\text{Total cost of manure} = \frac{12.50}{100} \times 1{,}500x_1 + \frac{12.50}{100} \times 1{,}800x_2 + \frac{12.50}{80} \times 1{,}200x_3$$

or $= 187.5x_1 + 225x_2 + 187.5x_3$

Labour cost:

Radish	:	$6x_1 \times 40 = 240x_1$
Muttar	:	$5x_2 \times 40 = 200x_2$
Potato	:	$6x_3 \times 40 = 240x_3$

Now,

Total Profit = Revenue – (Cost of Manure + Labour Cost)

$$= 5 \times 1{,}500x_1 + 4 \times 1{,}800x_2 + 5 \times 1{,}200x_3 - (187.5x_1 + 225x_2 + 187.5x_3 + 240x_1 + 200x_2 + 240x_3)$$
$$= 7072.5x_1 + 6775x_2 + 5572.5x_3$$

Now, with constraints on land availability and man-days availability, the LPP may be expressed as follows:

Maximise $Z = 7{,}072.5x_1 + 6{,}775.0x_2 + 5{,}572.5x_3$

Subject to

$x_1 + x_2 + x_3 \le 125$ (Land availability)

$6x_1 + 5x_2 + 6x_3 \le 500$ (Man-days availability)

$x_1, x_2, x_3 \ge 0$

Example 2.10 A manufacturing firm needs 5 component parts. Due to inadequate resources, the firm is unable to manufacture all its requirements. So the management is interested in determining as to how many, if any, units of each component should be purchased from outside and how many should be produced internally. The relevant data are given here.

Component	*M*	*A*	*T*	*TR*	*PP*	*PC*
C_1	4	1	1.5	20	48	30
C_2	3	3	2	50	80	52
C_3	1	1	0	45	24	18
C_4	3	1	0.5	70	42	31
C_5	2	0	0.5	40	28	16

M: Per unit milling time in hours

A: Per unit assembly time in hours

T: Per unit testing time in hours

TR: Total requirement in units

PP: Price per unit quoted in the market

PC: Per unit direct costs (including materials, labour, etc.)

Resources available are as follows:

Milling hours	:	300
Assembly hours	:	160
Testing hours	:	150

Formulate this as an LPP, taking the objective function as maximisation of saving by producing the components internally.

Let x_1, x_2, x_3, x_4, and x_5 represent the number of C_1, C_2, C_3, C_4, and C_5, respectively, produced internally.

Information Summary

Resources/ Constraints	*Component* C_1	C_2	C_3	C_4	C_5	*Total Availability*
Milling hrs (per unit)	4	3	1	3	2	300
Assembly hrs (per unit)	1	3	1	1	0	160
Testing hrs (per unit)	1.5	2	0	0.5	0.5	150
Requirement (units)	20	50	45	70	40	
Saving (per unit) (*PP-PC*)	18	28	6	11	12	

The problem would be:

Maximise $Z = 18x_1 + 28x_2 + 6x_3 + 11x_4 + 12x_5$

Subject to

$$\left.\begin{array}{l} 4x_1 + 3x_2 + x_3 + 3x_4 + 2x_5 \le 300 \\ x_1 + 3x_2 + x_3 + x_4 \le 160 \\ 1.5x_1 + 2x_2 + 0.5x_4 + 0.5x_5 \le 150 \end{array}\right\} \text{Capacity constraints}$$

$$\left.\begin{array}{l} x_1 \le 20 \\ x_2 \le 50 \\ x_3 \le 45 \\ x_4 \le 70 \\ x_5 \le 40 \end{array}\right\} \text{'Total requirements' constraints}$$

$x_1, x_2, x_3, x_4, x_5 \ge 0$ Non-negativity condition

Example 2.11 A firm produces three products *A*, *B* and *C*. It uses two types of raw materials I and II of which 5,000 and 7,500 units, respectively, are available. The raw material requirements per unit of the products are given below:

Raw Material	*Requirement per Unit of Product*		
	A	*B*	*C*
I	3	4	5
II	5	3	5

The labour time for each unit of product *A* is twice as that of product *B* and three times that of product *C*. The entire labour force of the firm can produce the equivalent of 3,000 units. The minimum demand for the three products is 600, 650 and 500 units respectively. Also, the ratio of the number of units produced must be equal to 2 : 3 : 4. Assuming the profits per unit of *A*, *B* and *C* are 50, 50 and 80, respectively, formulate the problem as a linear programming problem in order to determine the number of units of each product which will maximise the profit. *(CA, November, 1997)*

Let x_1, x_2 and x_3 be the output of products *A*, *B* and *C*, respectively. The LPP, using the given information, may be expressed as follows:

$$\text{Maximise} \quad Z = 50x_1 + 50x_2 + 80x_3 \qquad \text{Total Profit}$$

Subject to

$$\begin{aligned}
3x_1 + 4x_2 + 5x_3 &\le 5{,}000 && \text{Raw Material I}\\
5x_1 + 3x_2 + 5x_3 &\le 7{,}500 && \text{Raw Material II}\\
x_1 &\ge 600 && \\
x_2 &> 650 && \text{Minimum demand}\\
x_3 &\ge 500 && \\
x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &\le 3{,}000 && \text{Labour}\\
\frac{x_1}{2} = \frac{x_2}{3} &= \frac{x_3}{4} && \text{Output proportionality}\\
x_1, x_2, x_3 &\ge 0
\end{aligned}$$

Notes

1. In the absence of a statement to the contrary, it is assumed that availability of labour is sufficient to produce equivalent to 1,000 units of product *A*.
2. In a given case, if the output of two products *A* and *B*, is desired to be in a given ratio, say 2 : 3, we may express it as $3A = 2B$ or $A/2 = B/3$. For example, if the outputs are 10 and 15 (in the ratio 2 : 3), then we have $3 \times 10 = 2 \times 15$ or $10/2 = 15/3$. Accordingly, when three products have to be in the ratio 2 : 3 : 4, we write $x_1/2 = x_2/3 = x_3/4$.

Example 2.12 The Marketing Department of Everest Company has collected information on the problem of advertising for its products. This relates to the advertising media available, the number of families expected to be reached with each alternative, cost per advertisement, the maximum availability of each medium and the expected exposure of each one (measured as the relative value of one advertisement in each of the media):

The information is as given here:

Advertising Media	No. of Families Expected to Cover	Cost per Ad (Rs)	Maximum Availability (No. of times)	Expected Exposure (Units)
TV (30 sec)	3,000	8,000	8	80
Radio (15 sec)	7,000	3,000	30	20
Sunday edition of a daily (1/4 page)	5,000	4,000	4	50
Magazine (1 page)	2,000	3,000	2	60

Other information and requirements:

(a) The advertising budget is Rs 70,000.

(b) At least 40,000 families should be covered. (The families receiving messages could be common. But a family receiving three messages, for example, would be taken to be equivalent to three.)

(c) At least 2 insertions be given in Sunday edition of a daily but not more than 4 ads should be given on the TV.

Draft this as a linear programming problem. The company's objective is to maximise the expected exposure.

Let x_1 : the number of ads on TV,

x_2 : the number of ads on radio,

x_3 : the number of ads in Sunday edition of a daily, and

x_4 : the number of ads in a magazine.

Information Summary

Constraints/ Resources	Advertising Media				Total
	TV	Radio	Daily	Magazine	
No. of families (per ad)	3,000	7,000	5,000	2,000	40,000
Cost (per ad) and budget (Rs)	8,000	3,000	4,000	3,000	70,000
Max. availability (No.)	8	30	4	2	
Other information:					
	(i) at least two ads in daily				
	(ii) at most four ads on TV				
Expected Exposure (Units per ad)	80	20	50	60	

The problem would be as under:

Maximise $Z = 80x_1 + 20x_2 + 50x_3 + 60x_4$

Subject to

$$3{,}000x_1 + 7{,}000x_2 + 5{,}000x_3 + 2{,}000x_4 \geq 40{,}000$$

$$8{,}000x_1 + 3{,}000x_2 + 4{,}000x_3 + 3{,}000x_4 \leq 70{,}000$$

$$x_1 \leq 8$$

$$
\begin{aligned}
x_2 \qquad\qquad\qquad & \le 30 \\
x_3 \qquad\qquad & \le 4 \\
x_4 & \le 2 \\
x_3 \qquad\qquad & \ge 2 \\
x_1 \qquad\qquad\qquad\qquad & \le 4 \\
x_1, x_2, x_3, x_4 & \ge 0
\end{aligned}
$$

Example 2.13 A multinational company has two factories that ship to three regional warehouses. The costs of transportation per unit are:

Warehouse	Transportation Costs (Rs) Factory	
	F_1	F_2
W_1	2	4
W_2	2	2
W_3	5	3

Factory F_2 is old and has a variable manufacturing cost of Rs 20 per unit. Factory F_1 is modern and produces for Rs 10 per unit. Factory F_2 has a monthly capacity of 250 units, and Factory F_1 has a monthly capacity of 400 units. The requirements at the warehouses are:

Warehouse	Requirement
W_1	200
W_2	100
W_3	250

How should each factory ship to each warehouse in order to minimise the total cost? Formulate this problem as a linear programming model. Do not solve it. *(MBA, Delhi, April, 1996)*

The total cost (manufacturing plus transportation) matrix is given below:

Factory	Warehouse			Availability
	W_1	W_2	W_3	
F_1	12	12	15	400
F_2	24	22	23	250
Requirement	200	100	250	550/650

Let x_{ij} be the quantity shipped from ith factory to jth warehouse. The linear programming problem is;

Minimise $\quad Z = 12x_{11} + 12x_{12} + 15x_{13} + 24x_{21} + 22x_{22} + 23x_{23}$

Subject to

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} & \le 400 \\
x_{21} + x_{22} + x_{23} & \le 250 \\
x_{11} + x_{21} \qquad & = 200
\end{aligned}
$$

$$x_{12} + x_{22} = 100$$
$$x_{13} + x_{23} = 250$$
$$x_{ij} \geq 0 \qquad \text{for } i = 1, 2 \quad \text{and } j = 1,2,3$$

Example 2.14 The Orient Manufacturing Company produces three types of typewriters: Tik-Tik, Mik-Mik, and Pik-Pik. All the three types are required first to be machined and then to be assembled. The time requirements for the various types are as follows:

Type	*Machine Time* (in hours)	*Assembly Time* (in hours)
Tik-Tik	15	4.4
Mik-Mik	13	3.5
Pik-Pik	12	4.0

The total available machine time and assembly times are, respectively, 4,000 and 1,240 hours per month. The data regarding selling price, costs, and the contribution margin for the three are:

	Tik-Tik	*Mik-Mik*	*Pik-Pik*
Selling prices	Rs 11,000	Rs 5,000	Rs 3,000
Labour, material & other variable expenses	8,000	2,400	1,500
Contribution margin	3,000	2,600	1,500

The company sells all the three on one month credit basis, but labour, material and other variable expenses must be paid in cash. Some further information follows. The company has taken a loan of Rs 40,000 from a co-operative bank and has just been informed by the bank that the loan is not likely to be renewed when it expires at the beginning of the next month, viz. 1st January. The Good Bank of India, from whom the company has borrowed Rs 60,000, has expressed its willingness to renew the loan provided that the company maintains a quick ratio (i.e. the ratio of cash to the short term liabilities) of at least 1.

The balance sheet of the company as on 31st December is as follows:

Balance Sheet as on 31st December

Liabilities	*Rs*	*Assets*	*Rs*
Loan from Coop. Bank	40,000	Cash	200,000
Loan from Good		Receivables	50,000
Bank of India	60,000	Plant & Machinery	250,000
Long Term Debt	250,000		
Net Worth	150,000		
	500,000		500,000

It is further given that the company has to pay a fixed sum of Rs 10,000 each for interest on debt and for top management salaries every month.

Now, upon being informed about the co-operative bank's decision, the senior manager of the company requires this problem to be put as a linear programming problem. Assuming that (a) he wants to maximise the profits, and (b) he wants to maximise the total revenue from sales, do it for him.

Let x_1, x_2, x_3 be the number of units of Tik-Tik, Mik-Mik, and Pik-Pik produced.

Information Summary

Constraints/Resources	*Typewriter*			*Total*
	Tik-Tik	*Mik-Mik*	*Pik-Pik*	
Machine Hours	15/unit	13/unit	12/unit	4000
Assembly Hours	4.4/unit	3.5/unit	4.0/unit	1240
Cash (Rs) (for labour, materials and other various expenses)	8000	2400	1500	130,000*
Selling Price per unit (Rs)	11000	5000	3000	
Contribution Margin (Rs) per unit (Selling Price – Variable Exp.)	3000	2600	1500	

*Cash available is calculated as:

Cash Balance + Amt. from Receivables (to be received in January) – Co-operative Bank loan to be paid off – Cash required for debt interest and top management salaries – Cash required for matching the loan from Good Bank of India (which wants the quick ratio to be equal to 1)

$$= 200{,}000 + 50{,}000 - 40{,}000 - 20{,}000 - 60{,}000 = 130{,}000$$

(a) When the objective is to maximise profits:

Maximise $Z = 3{,}000x_1 + 2{,}600x_2 + 1{,}500x_3$

Subject to

$$\left.\begin{aligned} 15x_1 + 13x_2 + 12x_3 &\le 4{,}000 \\ 4.4x_1 + 3.5x_2 + 4.0x_3 &\le 1{,}240 \end{aligned}\right\} \text{Capacity constraints}$$

$$8000x_1 + 2400x_2 + 1500x_3 \le 130{,}000 \quad \text{Cash constraint}$$

$$x_1, x_2, x_3 \ge 0 \quad \text{Non-negativity condition}$$

(b) When the objective is to maximise total revenue, the objective function would be:

Maximise $Z = 11000x_1 + 5000x_2 + 3000x_3$

while all other constraints would be the same as shown in (a).

Example 2.15 A leading Chartered Accountant is attempting to determine a "best" investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.

Portfolio Data

	Shares under Consideration					
	A	*B*	*C*	*D*	*E*	*F*
Current Price per share (Rs)	80	100	160	120	150	200
Projected Annual Growth Rate	0.08	0.07	0.10	0.12	0.09	0.15
Projected Annual Dividend per Share (Rs)	4.00	4.50	7.50	5.50	5.75	0.00
Projected Risk in Return	0.05	0.03	0.10	0.20	0.06	0.08

The total amount available for investment is Rs 25 lakhs and the following conditions are required to be satisfied.

(i) The maximum rupee amount to be invested in alternative F is Rs 250,000.

(ii) No more than Rs 500,000 should be invested in alternatives A and B combined.

(iii) Total weighted risk should not be greater than 0.10 where

$$\text{Total Weighted Risk} = \frac{\text{(Amount Invested in Alternative } j\text{) (Risk of Alternative } j\text{)}}{\text{(Total Amount Invested in all the Alternatives)}}$$

(iv) For the sake of diversity, at least 100 shares of each stock should be purchased.

(v) At least 10 percent of the total investment should be in alternatives A and B combined.

(vi) Dividends for the year should be at least Rs 10,000.

Rupee return per share of stock is defined as price per share one year hence *less* current price per share *plus* dividend per share. If the objective is to maximise total rupee return, formulate the linear programming model for determining the optimal number of shares to be purchased in each of the shares under consideration. You may assume that the time horizon for the investment is one year. The formulated LP problem is not required to be solved. *(CA, November, 1991)*

Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number of shares of companies *A, B, C, D, E* and *F*, respectively. The rupee-return for various shares is shown as follows.

	Share					
	A	*B*	*C*	*D*	*E*	*F*
Projected growth per share (Rs)	6.40	7.00	16.00	14.40	13.50	30.00
Projected dividend	4.00	4.50	7.50	5.50	5.75	0.00
Rupee-return per share	10.40	11.50	23.50	19.90	19.25	30.00

Accordingly, the objective function is:

Maximise $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30.00x_6$

The constraints are as follows:

$$80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \leq 25{,}00{,}000 \quad \text{funds availability}$$

$$200x_6 \leq 250{,}000 \quad \text{condition (i)}$$

Total Weighted Risk

$$= \frac{(80x_1)(0.05) + (100x_2)(0.03) + (160x_3)(0.10) + (120x_4)(0.20) + (150x_5)(0.06) + (200x_6)(0.08)}{80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6} \le 0.10$$

On simplification, it gives

$$-4x_1 - 7x_2 + 12x_4 - 6x_5 - 4x_6 \le 0$$

From condition (iv), $x_i \ge 100$ for $i = 1, 2, 3, 4, 5, 6$

Condition (v) requires $80x_1 + 100x_2 \ge 0.10(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$

This becomes, $72x_1 + 90x_2 + 6x_3 - 12x_4 - 15x_5 - 20x_6 \ge 0$

The dividend requirement states that

$$4x_1 + 4.50x_2 + 7.50x_3 + 5.50x_4 + 5.75x_5 \ge 10{,}000$$

Accordingly, the LPP may be stated as follows:

Maximise $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30.00x_6$

Subject to

$$\begin{aligned}
80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 &\le 25{,}00{,}000 \\
200x_6 &\le 2{,}50{,}000 \\
80x_1 + 100x_2 &\le 5{,}00{,}000 \\
-4x_1 - 7x_2 + 12x_4 - 6x_5 - 4x_6 &\le 0 \\
72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 &\ge 0 \\
4x_1 + 4.5x_2 + 7.5x_3 + 5.50x_4 + 5.75x_5 &\ge 10{,}000 \\
x_1 &\ge 100 \\
x_2 &\ge 100 \\
x_3 &\ge 100 \\
x_4 &\ge 100 \\
x_5 &\ge 100 \\
x_6 &\ge 100
\end{aligned}$$

Example 2.16 A company engaged in producing tinned food delicacies has 300 trained employees on its rolls each of whom can produce one can of food in a week. Due to the developing taste of the public for this kind of food, the company plans to add the existing labour force by employing 150 people in a phased manner, over the next five weeks. The newcomers would have to undergo a two-week training programme before being put to work. The training is to be given by employees from amongst the existing ones and it is known that one employee can train three trainees. Assume that there would be no production coming forth from the trainers and the trainees during training period as the training is off-the-job. However, the trainees would be remunerated at the rate of Rs 300 per week, the same rate as is for the trainers.

The company has booked the following number of cans to supply during the next five weeks.

Week	:	1	2	3	4	5
No. of cans	:	280	298	305	360	400

Assume that the production in any week would not be more than the number of cans ordered for so that every delivery of the delicacy would be 'fresh'.

Using this information, draft a LPP to develop a training schedule that will minimise the labour costs over the five week period.

Let x_1, x_2, x_3, x_4, and x_5 be the number of trainees appointed in beginning of week 1, 2, 3, 4, and 5 respectively.

Information Summary

	Week				
	1	2	3	4	5
Cans Required	280	298	305	360	400
Other Information:					
(a) every trainee to be trained for 2 weeks					
(b) one employee required to train 3 trainees					
(c) every trained worker producing 1 can per week but no production from trainers and trainees during training					
(d) number of people to be employed = 150					
(e) the production in any week not to exceed the cans required					
No. of weeks for which newcomer would be employed*	5	4	3	2	1

*Thus, workers employed at the beginning of the first week would get salary for all 5 weeks under consideration, those employed at the beginning of the second week would get it for 4 weeks, and so on. The problem would be:

Minimise $$Z = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5$$

Subject to

$$\left.\begin{aligned}
300 - \tfrac{1}{3}x_1 &\geq 280\\
300 - \tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 &\geq 298\\
300 + x_1 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 &\geq 305\\
300 + x_1 + x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 &\geq 360\\
300 + x_1 + x_2 + x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 &\geq 400
\end{aligned}\right\} \text{Capacity constraints}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 150 \quad \text{New employment condition}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \quad \text{Non-negativity condition}$$

Notes (a) All along, the 300 employees, capable of producing 300 cans a week remain employed. The production capacity in different weeks (as shown on LHS of the constraints) is affected (negatively) by the fact that some of them are away for training and is affected (positively) by the fact that newcomers join the team of experienced workers.

(b) Inequalities have been used in the constraints because some workers might remain idle in some week(s).

(c) The training schedule as obtained by solving this problem would involve minimum cost in respect of the trainees. The cost would be obtained by multiplying the objective function by 300 because each person would get a salary of Rs 300 per week.

Example 2.17 Solve graphically:

Maximise $Z = 10x_1 + 15x_2$

Subject to

$$2x_1 + x_2 \le 26$$
$$2x_1 + 4x_2 \le 56$$
$$x_1 - x_2 \ge -5$$
$$x_1, x_2 \ge 0$$

This problem is exhibited graphically in Fig. 2.10. The feasible region is shaded. This is significant to note here that we have not considered the area below the line corresponding to $x_1 - x_2 = -5$ for the negative values of x_1. This is because of the non negativity condition $x_1 \ge 0$, which implies that negative values of x_1 should not be considered.

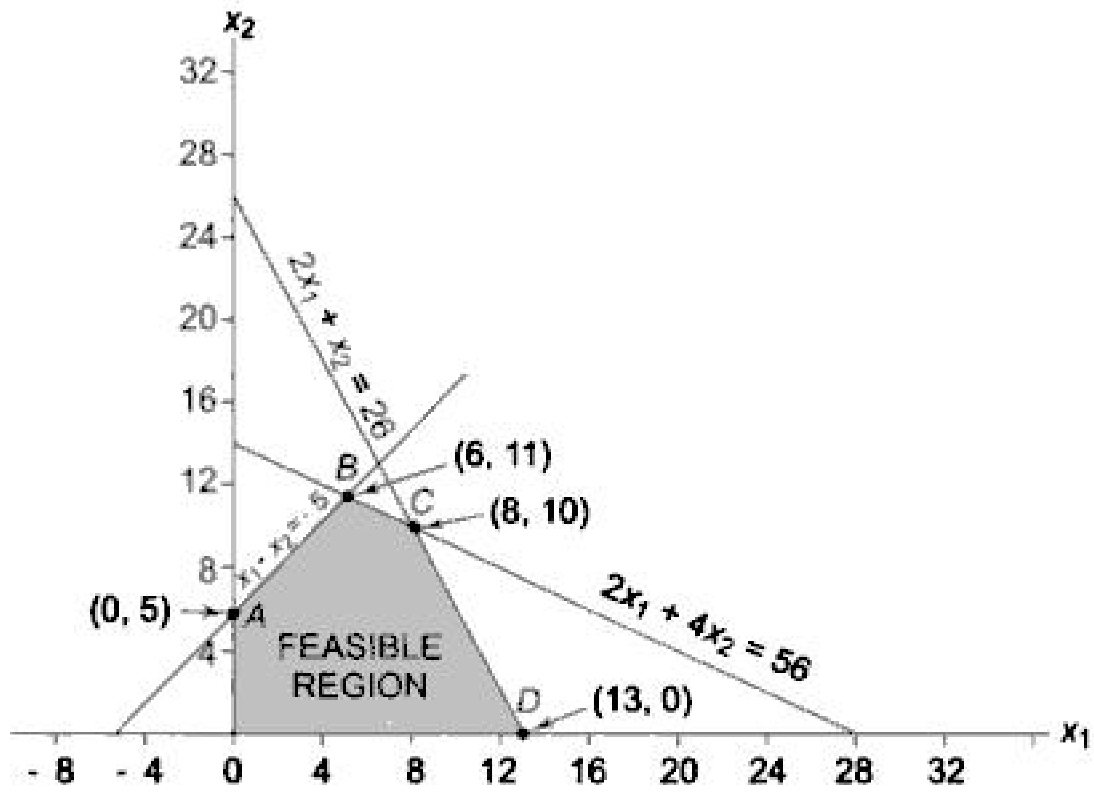


Fig. 2.10 Graphic Solution to the LPP

The Z-values at different points are given here:

Point	x_1	x_2	$Z = 10x_1 + 15x_2$
A	0	5	75
B	6	11	225
C	8	10	230
D	13	0	130

Thus Z is maximum (it is equal to 230) when $x_1 = 8$ and $x_2 = 10$.

Example 2.18 Solve graphically the following linear programming problem:

Minimise $Z = 3x_1 + 5x_2$

Subject to

$-3x_1 + 4x_2 \le 12$; $\quad 2x_1 - x_2 \ge -2$;

$2x_1 + 3x_2 \ge 12$; $\quad x_1 \le 4$; $x_2 \ge 2$, and

$x_1, x_2 \ge 0$ *(ICWA, June, 1985)*

The given problem is depicted graphically in Figure 2.11. The feasible region is shaded and is bounded by *ABCDE.*

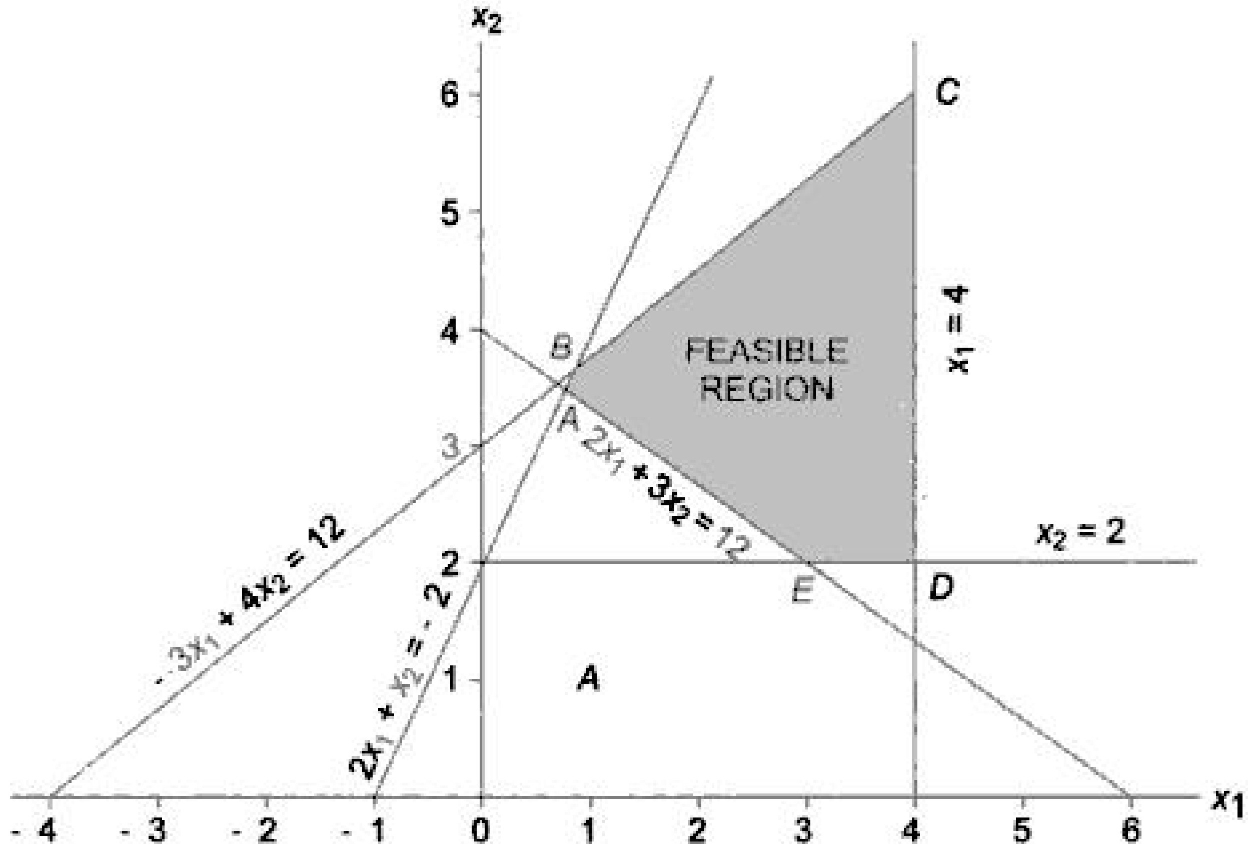


Fig. 2.11 Graphic Solution to the LPP

The different points are evaluated here:

Point	x_1	x_2	$Z = 3x_1 + 5x_2$
A	0.75	3.5	19.75
B	0.8	3.6	20.40
C	4.0	6.0	42.00
D	4.0	2.0	22.00
E	3.0	2.0	19.00*

Thus, Z is minimum at $x_1 = 3$ and $x_2 = 2$. Its value is 19.

Example 2.19 A retired person wants to invest upto an amount of Rs 30,000 in fixed income securities. His broker recommends investing in two bonds: Bond *A* yielding 7% and Bond *B* yielding 10%. After some consideration, he decides to invest at most Rs 12,000 in Bond *B* and at least Rs 6,000 in Bond *A*. He also wants the amount invested in Bond *A* to be at least equal to the amount invested in Bond *B*. What should the broker recommend if the investor wants to maximise his return on investment? Solve graphically.

(B Com (Hons), Delhi, 1999)

Let x_1 and x_2 be the amount invested in Bonds *A* and *B*, respectively. Using the given data, we may state the problem as follows:

Maximise $Z = 0.07x_1 + 0.10x_2$

Subject to

$$x_1 + x_2 \le 30{,}000$$
$$x_1 \ge 6{,}000$$
$$x_2 \le 12{,}000$$
$$x_1 - x_2 \ge 0$$
$$x_1, x_2 \ge 0$$

The constraints are plotted graphically in Figure 2.12. The feasible region is shown shaded and is bound by points A, B, C, D and E.

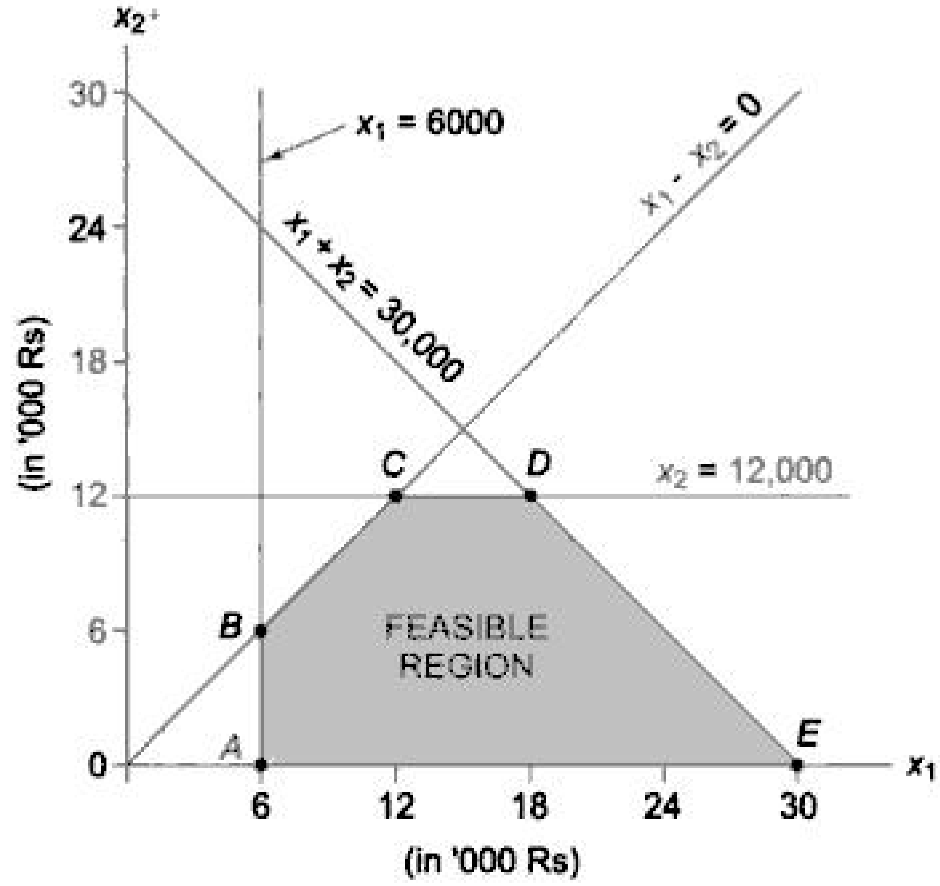


Fig. 2.12 Graphic Determination of Investment Mix

The extreme points are evaluated here.

Point	x_1	x_2	$Z = 0.07x_1 + 0.10x_2$
A	6,000	0	420
B	6,000	6,000	1,020
C	12,000	12,000	2,040
D	18,000	12,000	2,460
B	30,000	0	2,100

The Z-value is maximum at point D. Accordingly, the optimal solution is: invest Rs 18,000 in Bond A and Rs 12,000 in Bond B. It would yield a return of Rs 2,460.

Example 2.20 A local travel agent is planning a charter trip to a major sea port. The eight day and seven night package includes the fare for round trip, surface transportation, board and lodging and selected tour options. The charter trip is restricied to 200 persons and past experience indicates that there will not be any

problem for getting 200 clients. The problem for the travel agent is to determine the number of Deluxe, Standard and Economy tour packages to offer for this charter. These three plans differ according to seating and service for the flight, quality of accommodations, meal plans and tour options. The following table summarises the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an aircraft for the flat fee of Rs 2,00,000 for the entire trip.
In planning the trip, the following considerations must be taken into account:

(i) At least 10% of the packages must be of the Deluxe type.
(ii) At least 35% but not more than 70% must be of the Standard type.
(iii) At least 30% must be of the Economy type.
(iv) The maximum number of Deluxe packages available in any aircraft is restricted to 60.
(v) The hotel desires that at least 120 of the tourists should be on the Deluxe and Standard packages taken together.

Prices and Costs for Tour Packages per Person

Tour Plan	Price	Hotel Costs	Meal and Other Expenses
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

The travel agent wishes to determine the number of packages to offer in each type so as to maximise the total profit.

(a) Formulate this as a linear programming problem.
(b) Restate the above LPP in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.
(c) Find the optimal solution using graphical method for the restated problem and interpret your results.

(CA, May, 1991)

(a) Let x_1, x_2 and x_3 be the number of packages of Deluxe, Standard and Economy types, respectively. From the given information, unit profit for each of the three types can be obtained as under:

Deluxe Package : 10,000 – (3,000 + 4,750) = 2,250
Standard Package : 7,000 – (2,200 + 2,500) = 2,300
Economy Package : 6,500 – (1,900 + 2,200) = 2,400

With a hiring fee of Rs 2,00,000, the objective function can be expressed as:

Maximise $Z = 2{,}250x_1 + 2{,}300x_2 + 2{,}400x_3 - 2{,}00{,}000$

From the conditions given in the question, we have

$x_1 \geq 20$ condition (i)
$x_2 \geq 70$ and $x_2 \leq 140$ condition (ii)
$x_3 \geq 60$ condition (iii)

$$x_1 \le 60 \qquad \text{condition (iv) and}$$

$$x_1 + x_2 \ge 120 \qquad \text{condition (v)}$$

Also, $\quad x_1 + x_2 + x_3 = 200.$

Accordingly, the LPP can be stated as follows:

Maximise $\quad Z = 2250x_1 + 2300x_2 + 2400x_3 - 200{,}000$

Subject to

$$20 \le x_1 \le 60$$
$$70 \le x_2 \le 140$$
$$x_3 \ge 60$$
$$x_1 + x_2 \ge 120$$
$$x_1 + x_2 + x_3 = 200$$
$$x_1, x_2, x_3 \ge 0$$

(b) Since $x_1 + x_2 + x_3 = 200$, we have $x_3 = 200 - x_1 - x_2$. Substituting it in the above relations, the LPP can be re-expressed as follows:

Maximise $\quad Z = -150x_1 - 100x_2 + 280{,}000$

Subject to

$$20 \le x_1 \le 60$$
$$70 \le x_2 \le 140$$
$$120 \le x_1 + x_2 \le 140$$
$$x_1, x_2 \ge 0$$

(c) The problem is solved graphically as shown in Figure 2.13.

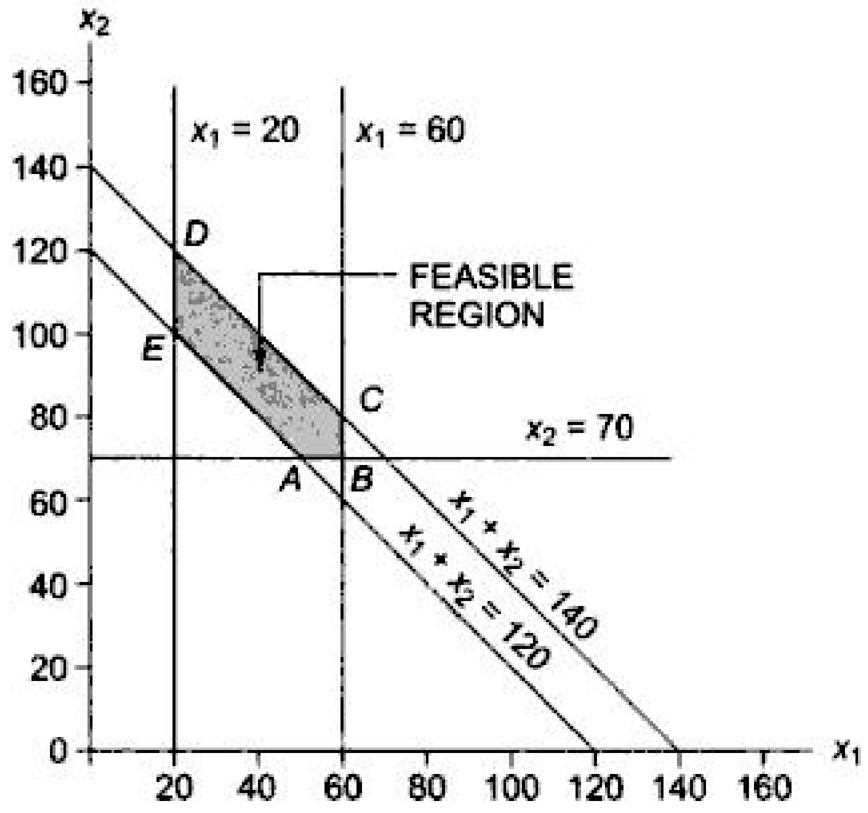


Fig. 2.13 Graphic Solution to the LPP

Point	x_1	x_2	$Z(= -150x_1 - 100x_2 + 280{,}000)$
A	50	70	265,500
B	60	70	264,000
C	60	80	263,000
D	20	120	265,000
E	20	100	267,000

Thus, maximum profit of Rs 267,000 can be achieved when $x_1 = 20$, $x_2 = 100$, and $x_3 = 80$.

Example 2.21 Let us assume that you have inherited Rs 100,000 from your father-in-law that can be invested in a combination of only two stock portfolios, with the maximum investment allowed in either portfolio set at Rs 75,000. The first portfolio has an average return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you want to maximise your return, you will not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Formulate this as a linear programming problem and solve it by graphic method. *(CA, May, 1999)*

Let x_1 : Investment in Portfolio 1

x_2 : Investment in Portfolio 2

The LPP may be expressed as under:

Maximise $Z = 0.10x_1 + 0.20x_2$ Total Return

Subject to

$x_1 + x_2 \leq 100{,}000$ Total Investment

$\left.\begin{aligned} x_1 &\leq 75000 \\ x_2 &\leq 75{,}000 \end{aligned}\right\}$ Investment Ceiling

$-2x_1 + 3x_2 \leq 0$ Risk Requirement

$-2x_1 + 8x_2 \geq 0$ Return Requirement

$x_1, x_2 \geq 0$ Non-negativity

Notes:

1. The average risk is not to be exceeding 6. Thus, we have,

$$\frac{4x_1 + 9x_2}{x_1 + x_2} \leq 6$$

or $4x_1 + 9x_2 \leq 6x_1 + 6x_2$

It simplifies to $-2x_1 + 3x_2 \leq 0$.

2. It is desired to have an average return of at least 12 percent. Thus,

$$0.10x_1 + 0.20x_2 \geq 0.12(x_1 + x_2),$$

which simplifies to

$$-2x_1 + 8x_2 \geq 0.$$

The given constraints are plotted in Figure 2.14.

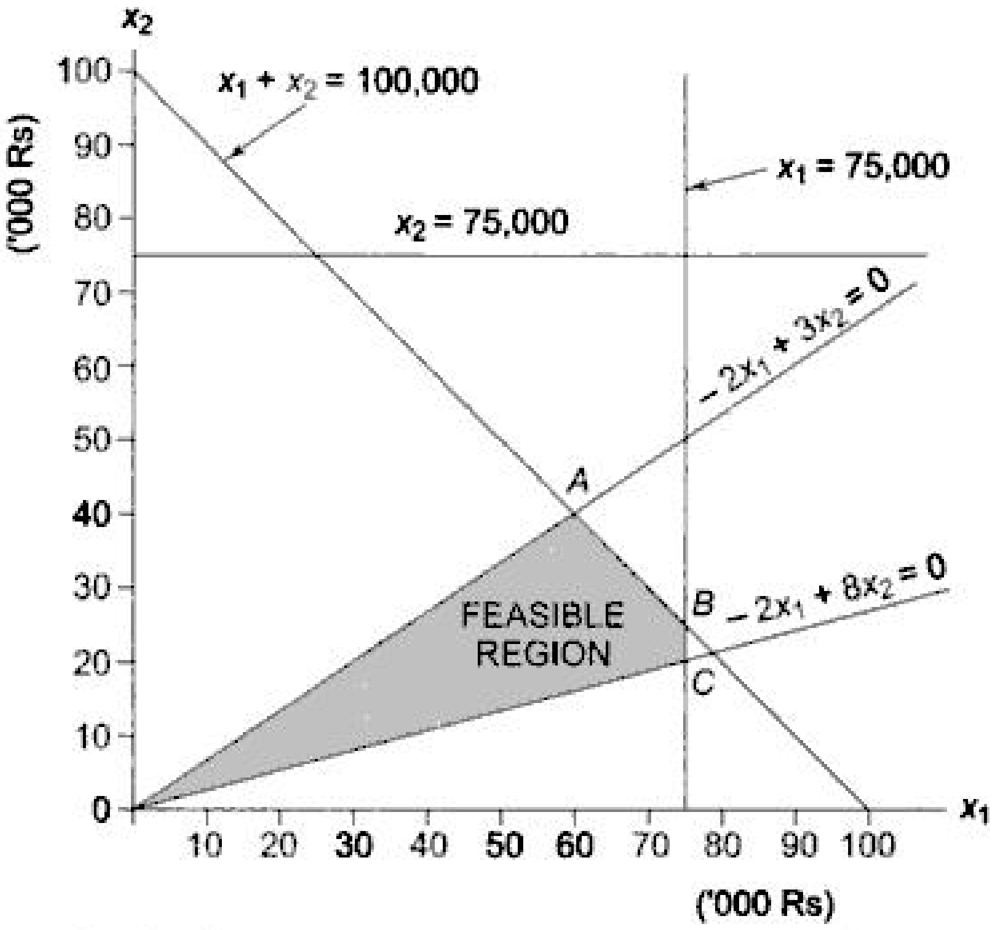


Fig. 2.14 Graphic Solution to Investment Problem

The feasible region is given by the shaded region with vertices *O*, *A*, *B* and *C*. These points are evaluated below.

Point	x_1	x_2	Z
0	0	0	0
A	60,000	40,000	14,000
B	75,000	25,000	12,500
C	75,000	18,750	11,250

Evidently, Rs 60,000 should be invested in Portfolio 1 and Rs 40,000 in Portfolio 2. It would yield a maximum return of Rs 14,000, while meeting all the given requirements.

Example 2.22 A manufacturer employs three inputs: man hours, machine-hours and cloth material to manufacture two types of dresses. Type *A* dress fetches him a profit of Rs 160 per piece, while type *B*, that of Rs 180 per piece. The manufacturer has enough man-hours to manufacture 50 pieces of type *A* or 20 pieces of type *B* dresses per day while the machine-hours he possesses suffice only for 30 pieces of type *A* or for 24 pieces for type *B* dresses. Cloth material available per day is limited but sufficient enough for 30 pieces of either type of drsss. Formulate the linear programming model and solve it graphically. *(M Com, Delhi, 2005)*

Let x_1 and x_2 be the number of dresses of types A and B respectively. Using the given information, the linear programming problem may be stated as follows:

Maximise $Z = 160x_1 + 180x_2$ Total profit

Subject to $\frac{x_1}{50} + \frac{x_2}{20} \le 1$ Man-hours

$\frac{x_1}{36} + \frac{x_2}{24} \le 1$ Machine-hours

$\frac{x_1}{30} + \frac{x_2}{30} \le 1$ Cloth material

$x_1, x_2 \ge 0$

The problem may be solved graphically now. The constraints are shown plotted in Fig. 2.15. The feasible solution is shown shaded and its vertices are evaluated below.

Point	x_1	x_2	Z
0	0	0	0
A	0	20	3,600
B	15	14	4,920
C	18	12	5,040
D	30	0	4,800

Thus, optimal solution calls for producing 18 units of type A dress and 12 units of type B dress. Total profit = Rs 5,040.

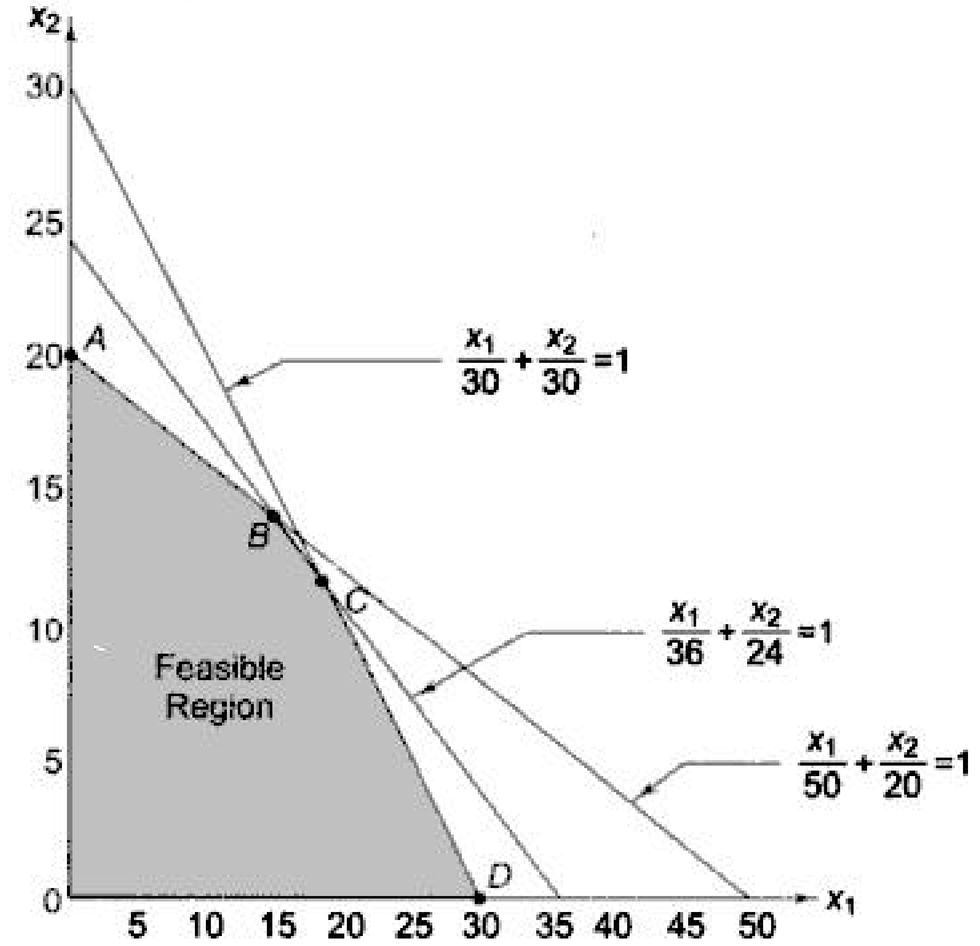


Fig. 2.15 Graphic Determination of Optimal Product Mix

KEY POINTS TO REMEMBER

- Mathematical programming involves optimisation of a certain function subject to certain constraints. An LPP is a mathematical programming problem which seeks to optimise (maximise or minimise) an objective function subject to a given set of constraints.
- Formulating an LPP requires identification of decision variables, setting up the objective function and the constraints, and examining and stating if the variables are non-negative or not.
- An LPP model is based on the assumptions of (i) proportionality, (ii) additivity, (iii) continuity of decision variables, (iv) certainty and (v) finite choices.
- Solution of an LPP graphically requires plotting all the constraints and determining the feasible region which ought to be a convex set. A constraint whose elimination does not affect feasible region is called a redundant constraint. The optimal solution may be found by evaluating the extreme points of the feasible region or through iso-profit/iso-cost lines.

- A problem might have a unique optimal solution, multiple optimal solutions, an unbounded solution or no feasible (and hence no optimal) solution.
- Multiple optimal solutions are obtained when the objective function is parallel to a constraint, which is binding and which forms an edge or boundary on the feasible region.
- Unbounded solution is present when the feasible region is unbounded from above and the objective function is of maximisation type, so that it is possible to increase the objective function value indefinitely.
- Infeasibility (no feasible solution) exists when there is no common point in the feasible areas for the constraints of a problem. The feasible region is empty in such a case.

TEST YOUR UNDERSTANDING

Mark the following statements as T (*True*) or F (*False*)

1 Basically, linear programming is a resource allocation problem that deals with the best allocation of limited resources to a number of competing activities.
2. A pre requisite for applying linear programming is that there should be an objective which is clearly identifiable and which may, or may not, be measurable in quantitative terms.
3. Proportionality property in the linear programming context implies that per unit contribution of a variable in the objective function is independent of the size of the variable.
4. The solution to an LPP implicitly assumes that the variables are continuous, which may take fractional as well as integer values in the solution.
5. All constraints in an LPP as well as its objective function must be linear in nature.
6. A typical linear programming problem is characterised by an objective function to be maximised or minimised, and a set of constraints, and non-negativity condition.
7. For an n variables linear programming problem, there must be an equal number of constraints.
8. An LPP can have only two decision variables.
9. Linear Programming is probabilistic in nature.
10. In real life, no variable can be unrestricted in sign.
11. An LPP must have all constraints of the "≤" or "≥" type.
12. A feasible solution is one which meets at least one of the constraints of the problem.
13. An optimal solution to a linear programming problem is a feasible solution which optimises.
14. The feasible region of a linear programming problem must be a convex set.
15. The graphic approach to the solution to LPPs cannot handle problems with more than three variables.
16. An iso-cost line cannot be parallel to the line of any constraint.
17. Iso-profit lines on a graph of an LPP would always be parallel to each other.
18. Replacing the '≤, sign of constraint by a '=' sign would improve the value of the objective function.
19. The feasible region for a constraint is restricted if its '≥, or '≤, sign is replaced by a '=' sign.
20. A constraint $4x_1 + 7x_2 \geq 57$ of an LPP is replaced by the constraint $4x_1 + 7x_2 \geq 40$. This would make the LPP more restrictive in nature.
21. Exclusion of a redundant constraint does not affect the optimal solution to an LPP. Thus, a redundant constraint represents an abundant resource.

22. A linear programming problem cannot have more than one redundant constraint.
23. For a linear programming model, the feasible region may change if non-binding constraints are deleted.
24. Every linear programming problem has a unique optimal solution.
25. It is possible for the objective function value of an LPP to be the same at two distinct extreme points.
26. When infeasibility does not exist, it is always possible to determine the optimal solution from a knowledge of all the extreme points of the polygon of the feasible region.
27. Changes in the objective function coefficients shall always result in changing the optimal values of the decision variables.
28. Infeasibility indicates that there are very few feasible solutions to an LPP and it is, therefore, difficult to say which of these is optimal.
29. An LPP with an unbounded feasible region would obviously have unbounded solution.
30. For a linear programming problem to be unbounded, its feasible region must be unbounded.

EXERCISES

1. What is a linear programming problem? Discuss the scope and role of linear programming in solving management problems.
2. Discuss and describe the role of linear programming in managerial decision-making bringing out limitations, if any. *(MBA, Delhi, 1999)*
3. "Linear Programming is one of the most frequently and successfully used Operations Research technique to managerial and business decisions." Elucidate this statement with some examples.
4. Give the mathematical and economic structure of the linear programming problem. What requirements should be met in order that the linear programming may be applied?
5. Briefly explain the major applications of linear programming in business.
6. What are the components of an LPP? What does the non-negativity restriction mean?
7. Give a general statement of a linear programming problem. Is it correct to say that the constraints should be of 'less than or equal to' form for the maximisation problems and of 'more than or equal to' form for the minimisation problems?
8. Discuss the assumptions of proportionality, additivity, continuity, certainty and finite choices in the context of LPPs.
9. In relation to linear programming, explain the implications of the following assumptions of the model:
   (a) linearity of objective function and constraints,
   (b) continuous variable,
   (c) certainty.
10. What steps are required in solving LPPs by graphic method? Discuss in brief.
11. What is feasibility region? Is it necessary that it should always be a convex set?
12. Define iso-profit line. How does it help to obtain solution to the linear programming problems?
13. What is a redundant constraint? What does it imply? Does it affect the optimal solution to an LPP?
14. How would you know whether the solution to a linear programming problem is unique or not? In this connection, state the conditions that should be satisfied for more than one optimal solution to a problem to exist.

15. Explain the phenomenon of 'infeasibility' in an LP problem. What are the indicators of such a phenomenon? How can it be handled? Write a problem which you think will not have a feasible solution.
(M Com, Delhi, 1999)

Practical Problems

1. Consider the production planning of The Super Fast Manufacturing Company which makes items *P* and *V*. The steel requirement for *P* is 400 gm per piece and that for *V* is 350 gm per piece. Both *P* and *V*, are machined on lathe which takes 85 and 50 minutes respectively, and are processed on a grinder which requires 55 and 30 minutes respectively. Each unit of *P* consumes 20 minutes of polishing time. The resource availability is:

   Total Machine Time : 1, 450 hours

   Total Steel : 250 kg

   30 per cent of total machine time is that of lathe, 50 per cent of grinder and the remaining of polishing. Unit contribution to profits for *P* and *V* is Rs 40 and Rs 30, respectively.

   Formulate this as a linear programming model for determining the number of units of *P* and *V* to be produced which would maximise the profits. Given also is the constraint that the company cannot sell more units of item *P* than of item *V*.

2. A company manufactures 3 types of parts which use precious metals platinum and gold. Due to shortage of these precious metals, the government regulates the amount that may be used per day. The relevant data with respect to supply, requirements, and profits are summarised in the table as follows:

Product	*Platinum required/unit (gms)*	*Gold required/unit (gms)*	*Profit/unit (Rs)*
A	2	3	500
B	4	2	600
C	6	4	1,200

   Daily allotment of platinum and gold are 160 gm and 120 gm respectively. How should the company divide the supply of scarce precious metals?

   Formulate it as a linear programming problem.

3. A manufacturer of purses makes four styles of purses: a three-compartment bag which takes 45 minutes to assemble; a shoulder-strap bag, taking one hour to assemble; a tote bag, needing 45 minutes for assembly, and pocket purse requiring 30 minutes to assemble. There are 32 hours of assembly time available per day. The profit contribution on the sale of a three-compartment bag is Rs 16, Rs 25 on a shoulder-strap bag, and Rs 12 each on tote bag and pocket purse.

   Special kind of fancy pins are used in decorating pocket purses and they are available for only 30 pieces. Different type of pins are used in other three types of bags of which only 70 are in stock. Enough raw material is available for a total of 60 pocket purses and tote bags which need same quantity of raw material. The manufacturer estimates a minimum demand of 6 pocket purses and 10 shoulder strap bags every day.

   Formulate a linear programming problem to optimise daily production.

4. An electronics company is engaged in the production of two components C_1 and C_2, used in radio sets. Each unit of C_1 costs the company Rs 5 in wages and Rs 5 in materials, while each unit of C_2 costs the company Rs 25 in wages and Rs 15 in materials. The company sells both products on one-period credit

terms, but the company's labour and material expenses must be paid in cash. The selling price of C_1 is Rs 30 per unit and of C_2 it is Rs 70. Because of the strong monopoly of the company for these components, it is assumed that the company can sell at the prevailing prices as many units as it produces. The company's production capacity is, however, limited by two considerations. First, at the beginning of period 1, the company has initial balance of Rs 4,000 (cash *plus* bank credit *plus* collections from past credit sales). Second, the company has available, in each period, 2,000 hours of machine time and 1,400 hours of assembly time. The production of each C_1 requires 3 hours of machine time and 2 hours of assembly time, whereas the production of each C_2 requires 2 hours of machine time and three hours of assembly time.

Formulate the above problem as a linear programming problem.

5. At the beginning of a month, a lady has Rs 30,000 available in cash. She expects to receive certain revenues at the beginning of the months 1, 2, 3 and 4 and pay the bills after that, as detailed here:

Month	*Revenue*	*Bills*
1	Rs 28,000	Rs 36,000
2	Rs 52,000	Rs 31,000
3	Rs 24,000	Rs 40,000
4	Rs 22,000	Rs 20,000

It is given that any money left over may be invested for one month at the interest rate of 0.5%; for two months at 1.0% per month; for three months at 1.5% per month and for four months at 1.8% per month. Formulate her problem as a linear programming problem to determine an investment strategy that maximises cash in hand at the beginning of the month 5.

6. Command Area Development authority in the command of River X desires to find out the optimal cropping pattern in the area. Total available land is 25 thousand acres. The following crops can be grown:

	Water consumption in acre feet/acre	*Expected profit per acre (in Rs)*
Wheat	9	2,000
Maize	6	1,500
Jowar	6.5	1,200

It is felt that we cannot use more than 50% of the available land for wheat. Available water is 50,000 acre feet. At least 20% of land must be devoted to maize. To ensure balanced development of various crops, the ratio of land devoted to wheat and jowar should not be more than 3 : 2.

Formulate the above as a Linear Programming Problem to maximise total profit.

(Grad. Prg. in OR, OSRI, May, 1983)

7. A Mutual Fund company has Rs 20 lakhs available for investment in Government bonds, blue chip stocks, speculative stocks and short-term bank deposits. The annual expected return and risk factors are given as follows:

Type of investment	*Annual expected return*	*Risk factor (0 to 100)*
Government Bonds	14%	12
Blue Chip Stocks	19%	24
Speculative Stocks	23%	48
Short-term Deposits	12%	6

Mutual Fund is required to keep at least Rs 2 lakhs in short-term deposits and not to exceed an average risk factor of 42. Speculative stocks must be at most 20 percent of the total amount invested. How should Mutual Fund invest the funds so as to maximise its total expected annual return? Formulate this as a Linear Programming Problem. Do not solve it. *(CA, May, 1996)*

8. A company produces three types of parts for automatic washing machines. It purchases castings of the parts from a local foundry and then finishes the parts on drilling, shaping, and polishing machines. The selling prices of parts *A*, *B* and *C* respectively, are Rs 8, Rs 10 and Rs 14. All parts made can be sold. Castings for parts *A*, *B* and *C*, respectively cost Rs 5, 6 and 10. The company possesses only one of each type of machine. Costs per hour to run each of the three machines are Rs 20 for drilling, Rs 30 for shaping and Rs 30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the following table:

Machine	Capacity per hour		
	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The manager of the company wants to know how many parts of each type to produce per hour in order to maximise profit for the hour's run. Formulate the above as a linear programming problem. *(MBA, Delhi, October, 1997)*

9. A city hospital has the following minimal daily requirements for nurses:

Period	Clock time (24 hour day)	Minimal number of nurses required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m. – 6 a.m.	6

Nurses report at the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be a sufficient number of nurses available for each period.

Formulate this as a linear programming problem by setting up appropriate constraints and objective function. Do not solve. *(B Com (Hons), Delhi, 2006)*

10. Evening shift resident doctors in the B' Healthy Hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working:

S	M	T	W	T	F	S
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate a general linear programming model to minimise the number of doctors employed by the hospital.

11. Four products have to be processed through the plant, the quantities required for the next production period being:

Product 1	2,000 units
Product 2	3,000 units
Product 3	3,000 units
Product 4	6,000 units

There are three production lines on which the products could be processed. The rates of production in units per day and the total available capacity in days are given in the following table. The cost of using the lines is Rs 600, Rs 500 and Rs 400 per day, respectively:

Production line	*Product* 1	2	3	4	*Maximum line Capacity (days)*
1	150	100	500	400	20
2	200	100	760	400	20
3	160	80	890	600	18
Total	2,000	3,000	3,000	6,000	

Formulate as a linear programming problem to minimise the cost of operation.

12. A certain firm has two plants. Orders from four customers have been received. The number of units ordered by each customer and the shipping cost from each plant are shown in the following table:

Customer	*Units ordered*	*Shipping cost (Rs)/unit*	
		From plant 1	*From plant 2*
A	500	15	40
B	300	20	30
C	1,000	30	25
D	200	35	20

Each unit of the product must be machined and assembled. These costs, together with the capacities at each plant, are shown below:

	Hours/Unit	***Cost(Rs)/Hour***	***Hours available***
Plant No 1:			
Machining	0.10	40	120
Assembling	0.20	30	260
Plant No.2:			
Machining	0.11	40	140
Assembling	0.22	30	250

Formulate a linear programming problem to minimise cost. *(MBA, Delhi, December, 1993)*

13. Formulate the following as a linear programming problem. Do not solve.

    A trucking company with Rs 40,00,000 to spend on new equipment is contemplating three types of vehicles. Vehicle *A* has a 10-tonne pay-load and is expected to average 35 km per hour. It costs Rs 80,000. Vehicle *B* has a 20-tonne pay-load and is expected to average 30 km per hour. It costs Rs 1,30,000. Vehicle *C* is a modified form of *B*; it carries sleeping quarter for one driver, and this reduces its capacity to 18 tonnes and raises the cost to Rs 1,50,000. Vehicle *A* requires a crew of one man, and if driven on three shifts per day, could be run for an average of 18 hours per day. Vehicles *B* and *C* require a crew of two men each, but whereas *B* would be driven 18 hours per day with three shifts, *C* could average 21 hours per day. The company has 150 drivers available each day and would find it very difficult to obtain further crews. Maintenance facilities are such that the total number of vehicles must not exceed 30. How many vehicles of each type should be purchased if the company wishes to maximise its capacity in tonne-kms per day? *(MBA, Delhi, November, 1995)*

14. A refinery makes three grades of petrol *A, B, C* from three crude oils *d, e* and *f*. Crude oil *f* can be used in any grade but the others must satisfy the following specifications:

Grade	Selling price per liter	Specification
A	18.0	Not less than 50% crude *d*
		Not more than 25% crude *e*
B	16.5	Not less than 25% crude *d*
		Not more than 50% crude *e*
C	15.5	No specifications

There are capacity limitations on the amounts of the three crude elements that can be used:

Crude	Capacity (kl)	Price per liter
d	500	19.5
e	500	14.5
f	360	15.1

It is desired to obtain maximum profit. Formulate this as a Linear Programming Problem.

15. A company wants to plan production for the ensuing year so as to minimise the combined cost of production and inventory storage costs. In each quarter of the year, demand is anticipated to be 65, 80, 135 and 75 respectively. The product can be manufactured during regular time at a cost of Rs 16 per unit produced, or during overtime at a cost of Rs 20 per unit. The table given below gives data pertinent to production capacities. The cost of carrying one unit in inventory per quarter is Rs 2. The inventory level at the beginning of the first quarter is zero.

Quarter	Capacities (units)		Quarterly demand
	Regular time	Overtime	
1	80	10	65
2	90	10	80
3	95	20	135
4	70	10	75

Formulate the given problem to minimise the production plus storage costs for the year.

(MBA, Delhi, March, 2004)

16. Consider a company that must produce two products over a production period of three months of duration. The company can pay for the materials and labour from two sources: company funds and borrowed funds. The company faces three decisions:

(1) How many units should it produce of Product 1?

(2) How many units should it produce of Product 2?

(3) How much money should it borrow to support the production of the two products?

In making these decisions, the company wishes to maximise the profit contribution subject to the conditions given below:

(i) Since the company's products are enjoying a seller's market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints together with cost and price data are given in the table below.

Capacity, Price and Cost Data

	Selling price	Cost of production	Required hours per unit in department		
Product	(Rs per Unit)	(Rs per Unit)	A	B	C
1	14	10	0.5	0.3	0.2
2	11	8	0.3	0.4	0.1
Available hours per production period of three months			500	400	200

(ii) The available company funds during the production period will be Rs 3 lakhs.

(iii) A bank will give loans upto Rs 2 lakhs per production period at an interest rate of 20 % per annum provided the company's acid (quick) test ratio is at least equal to 1 to 1 while the loan is outstanding. Take a simplified acid-test ratio given by

$$\frac{\text{Surplus cash on hand after production + Accounts receivable}}{\text{Bank borrowing + Interest accrued thereon}}$$

(iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a Linear Programming Problem. *(CA, November, 1992)*

17. A manufacturer currently produces four products. Recent recessionary trends cause a decline in demand and the company is laying off workers and discontinuing its third shift.

The problem is that of rescheduling production during the first and second shifts for the remaining quarter of the year. The production involves various processes and one limiting resource in production is the availability of machine hours for a particular process Z. For this process, the four products require 4, 5, 5, and 7 hours, respectively.

The sales manager has forecast the expected sales for each of the four products in the last quarter of the year. The estimates are shown in the following table:

Month	Forecast sales			
	Product #1	Product #2	Product #3	Product #4
October	8,000	19,000	4,000	7,000
November	7,000	19,000	15,000	7,000
December	6,000	18,000	17,000	7,000

The production capacity in terms of process Z, hours available, is expressed by month and shift.

Month	Process Z, hours available	
	Shift 1	Shift 2
October	110,000	100,000
November	130,000	120,000
December	115,000	116,000

The labour cost of operating the process Z machines is Rs 100 per hour during the first shift and Rs 120 for the second shift. The other relevant cost is storage. It costs Rs 40 per month to store one unit of any of the four products. It may be noted that it will be necessary to store some units of the four products as there is not enough labour available during the December demand.

Assuming that the company wishes to produce as many products as the sales manager has forecast, formulate an LP model to determine a production schedule that will meet the demand at minimum cost.

(Grad.Prg. in OR, ORSI, November 1983)

18. A manufacturer receives an order from a state transport corporation for six buses, to be delivered two at a time over the next three months. Production data for the manufacturer are shown in the following table:

	Months		
	1	2	3
Regular production capacity (in units)	1	2	3
Overtime production capacity (in units)	2	2	2
Regular production cost (Rs 10,000/unit)	35	43	40
Overtime production cost (Rs 10,000/unit)	39	47	45

Buses can be delivered to the city at the end of the same month in which they are assembled, or they can be stored by the manufacturer, at a cost of Rs 3,000 per bus per month for shipment during a later month. The manufacturer has no current inventory of these buses and desires none after the completion of this contract.

Formulate this problem as a linear programming problem to determine a production schedule that will meet the corporation demand at minimum cost to the manufacturer. *(MBA, Delhi, 1986)*

19. WELLTYPE Manufacturing Company produces three types of typewriters: Manual typewriters, Electronic typewriters, and Deluxe Electronic typewriters. All the three models are required to be machined first and then assembled. The times required for the various models are as follows:

Type	Machine time (in hours)	Assembly time (in hours)
Manual typewriter	15	4
Electronic typewriter	12	3
Deluxe Electronic typewriter	14	5

The total available machine time and assembly time are 3,000 hours and 1,200 hours, respectively. The data regarding the selling price and variable costs for the three types are:

	Manual	Electronic	Deluxe
Selling Price (Rs)	4,100	7,500	14,600
Labour, material and other variable costs(Rs)	2,500	4,500	9,000

The company sells all the three on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has a loan of Rs 40,000 from a co-operative bank and this company will have to repay it to the bank on 1st April, 1993. The TNC Bank from whom this company has borrowed Rs 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this company as on 31.3.93 is as follows:

Liabilities	Rs	Assets	Rs
Equity Share Capital	150,000	Land	90,000
Capital Reserve	15,000	Building	70,000
General Reserve	110,000	Plant & Machinery	100,000
Profit & Loss a/c	25,000	Furniture & Fixtures	15,000
Long term loan	100,000	Vehicles	30,000
Loan from TNC Bank	60,000	Inventory	5,000
Loan from Co-op. Bank	40,000	Receivables	50,000
		Cash	140,000
Total	500,000	Total	500,000

The company will have to pay a sum of Rs 10,000 towards the salary for top management executives and other fixed overheads for the month. Interest on long-term loans is to be paid every month at 24% per annum. Interest on loans from TNC and Co-operative Banks may be taken to be Rs 1,200 for the month. Also, this company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic typewriters to one of its valued customers next month.

Also, make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all three types of typewriters in the market. The Senior

Manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximise the profits of the company.

Formulate this as a linear programming problem. The formulated problem need not be solved.

(CA, May, 1993)

20. Obtain graphically the solution to the following LPP:

Maximise $Z = x_1 + 3x_2$

Subject to

$$x_1 + 2x_2 \leq 9$$
$$x_1 + 4x_2 \leq 11$$
$$x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

21. Find solution to the following LP problem:

Maximise $Z = 10x_1 + 8x_2$

Subject to

$$2x_1 + x_2 \leq 20$$
$$x_1 + 3x_2 \leq 30$$
$$x_1 - 2x_2 \geq -15$$
$$x_1, x_2 \geq 0$$

22. A firm makes two products *X* and *Y*, and has a total production capacity of 9 tonnes per day, *X* and *Y* requiring the same production capacity. The firm has a permanent contract to supply as least 2 tonnes of *X* and at least 3 tonnes of *Y* per day to another company. Each tonne of *X* requires 20 machine hours production time and each tonne of *Y* requires 50 machine hours production time. The daily maximum possible number of machine-hours is 360. All the firm's output can be sold, and the profit made is Rs 80 per tonne of *X* and 120 per tonne of *Y*. It is required to determine the production schedule for maximum profit and to calculate this profit. Use Graphical Method to get your solution.

(MBA, Delhi, April, 1996)

23. Cashewco has two grades of cashew nuts: Grade I—750 kg and Grade II—1,200 kg. These are to be mixed in two types of packages of one kilogram each—economy and special. The economy pack consists of grade I and grade II cashews in the proportion of 1 : 3, while the special pack combines the two in equal proportion. The profit margin on the economy and special packs is, respectively, Rs 5 and Rs 8 a pack.

    (a) Formulate this as a linear programming problem.

    (b) Ascertain graphically the number of packages of economy and special types to be made as will maximise the profits.

    Would your answer be different if the profit margin on a special pack be Rs 10?

24. A company produces two types of pencils, say A and B. Pencil A is of superior quality and pencil B is of lower quality. Profits on the pencils A and B are Rs 5 and 3 per pencil respectively. Raw material required for each pencil A is twice as much as that of pencil B. The supply of raw material is sufficient only for 1000 pencils of type B per day. Pencil A requires a special clip and only 400 clips are available per day. For Pencil B only 700 clips are available per day. Use graphical method to find the product-mix that the company can make to make maximum profit. *(MBA, Delhi, March 2004)*

25. A company has two grades of inspectors 1 and 2, who are to be assigned for a quality control inspection. It is required that at least 2,000 pieces be inspected per 8- hour day. A grade 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97 percent. A grade 2 inspector can check at the rate of 30 pieces per hour, with an accuracy of 95 percent.

    The wage rate of grade 1 inspector is Rs 5 per hour while that of a grade 2 inspector is Rs 4 per hour. An error made by inspector costs Rs 3 per piece. There are only 9 grade 1 inspectors and 11 grade 2 inspectors available in the company. The company wishes to assign work to the available inspectors so as to minimise the total cost of inspection. Formulate this problem as a linear programming model and solve it by using graphical method. *(MBA, Delhi, March 2004)*

26. Attempt graphically the following problem:

    Maximise $3x_1 + 2x_2$

    Subject to

    $$2x_1 + x_2 \le 12;\ x_1 + x_2 \le 10;\ -x_1 + 3x_2 \ge 6;$$

    and $x_1, x_2 \ge 0$

27. Minimise $Z = 3x_1 + 10x_2$

    Subject to

    $$15x_1 + 4x_2 \ge 60;\ 8x_1 + 8x_2 \ge 40;\ 4x_1 + 16x_2 \ge 32$$

    and further that both the variables are non-negative.

28. Solve graphically the following LPP:

    Minimise $Z = 4x_1 + 3x_2$

    Subject to

    $$x_1 + 3x_2 \ge 9$$
    $$2x_1 + 3x_2 \ge 12$$
    $$x_1 + x_2 \ge 5$$
    $$x_1, x_2 \ge 0$$

29. A company manufactures two kinds of machines, each requiring a different manufacturing technique. The deluxe machine requires 18 hours of labour, 9 hours of testing, and yields a profit of Rs 400. The standard machine requires 3 hours of labour, 4 hours of testing, and yields a profit of Rs 200. There are 800 hours of labour and 600 hours of testing available each month. A marketing forecast has shown the monthly demand for the standard machine to be no more than 150. The management wants to know the numbers of each model to produce monthly that will maximise total profit. Formulate and solve this as a linear programming problem. *(MBA, Delhi, November, 2003)*

30. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. Because of the need to ensure certain nutrient constituents, it is necessary to buy one or two additional products, called *A* and *B*.

    The nutrient constituent (vitamins and proteins) in each unit of the products are as follows:

Nutrient	Nutrient contents in the product		Min. nutrient amount
	A	B	
1	36	6	108
2	3	12	36
3	20	10	100

Product *A* costs Rs 20 and product *B* costs Rs 40 per unit. How much of products *A* and *B* should be purchased at the lowest possible cost so as to provide the pigs nutrients not less than the minimum required as given in the table.

31. Two kinds of food for children, F_1 and F_2, are being considered to be purchased. Food F_1 costs Rs 20 a unit while food F_2 is available at Rs 40 per unit. The nutrient contents of these foods are as follows:

Nutrients	Nutrient content	
	Food F_1	Food F_2
N_1	40	20
N_2	3	12
N_3	18	3

The minimum requirement of three nutrients is respectively, 200, 36 and 54 units.

Draft this as an LPP and find out graphically the quantities of food units which should be bought in order that the costs are minimised satisfying the given constraints.

32. (a) Is it necessary that the feasible region for a maximisation type of linear programming problem must always be a bounded one?

(b) Maximise $Z = 6x_1 - 2x_2,$
Subject to

$$8x_1 - 4x_2 \le 8$$
$$0 \le x_1 \le 3$$
$$x_2 \ge 0$$

Solve graphically.

33. A medicine manufacturing company plans to market two syrups: Tonus-2,000, and Health-Wealth. There are sufficient ingredients available to make 20,000 bottles of Tonus-2,000 and 40,000 bottles of Health-Wealth. It takes three hours to prepare enough material to fill 1,000 bottles of Tonus-2,000 and one hour to fill 1,000 bottles of Health-Wealth. Only 45,000 bottles and 66 hours are available for this operation next week. The profit expected is Rs 2.80 per bottle of Tonus-2000 and Rs 2.20 per bottle of Health-Wealth.

Formulate this problem as an LPP and determine graphically the optimum production programme.

34. A cold drinks company has two bottling plants, located at different places. Each plant produces three different drinks *A*, *B* and *C*. The capacities of the two plants in number of bottles per day are as follows:

	Product A	Product B	Product C
Plant I	3,000	1,000	2,000
Plant II	1,000	1,000	6,000

A market survey indicates that during any particular month there will be a demand of 24,000 bottles of *A*, 16,000 bottles of *B* and 48,000 bottles of *C*. The operating costs, per day, of running plants I and II are, respectively, Rs 6,000 and Rs 4,000. How many days should the company run each plant during the month so that the production cost is minimised while still meeting the market demand? Use graphical method to get the optimal solution. *(PGDB and M in SM, Delhi, 1987)*

35. A firm is engaged in producing two products: P_1 and P_2.
The relevant data are given here:

	Per Unit:	Product P_1	Product P_2
(i)	Selling price	Rs 200	Rs 240
(ii)	Direct materials	Rs 45	Rs 50
(iii)	Direct wages:		
	Deptt A	8 hrs @ Rs 2/hr	10 hrs @ Rs 2/hr
	Deptt B	10 hrs @ Rs 2.25/hr	6 hrs @ Rs 2.25/hr
	Deptt C	4 hrs @ Rs 2.5/hr	12 hrs @ Rs 2.5/hr
(iv)	Variable overheads	Rs 6.50	Rs 11.50

Fixed overhead = Rs 2,85,000 per annum.
No. of employees in the three departments: Deptt A = 20; Deptt B = 15; Deptt C = 18
No. of hours/employee/week = 40 in each department
No. of weeks per annum = 50.

(a) Formulate the given problem as a linear programming problem, and solve graphically to determine
 (i) the product mix as will maximise the contribution margin of the firm.
 (ii) the amount of contribution margin and profit obtainable per year.

(b) From the graph, do you observe any constraint that is redundant? Which one, if yes?

36. Solve graphically the following LPP:

Minimise $Z = -4x_1 + 3x_2$

Subject to

$$x_1 - 2x_2 \geq -4;\ 2x_1 + 3x_2 \geq 13;\ x_1 - x_2 \geq 4;\text{ and } x_1, x_2 \geq 0$$

37. Subject to

$$2x_1 - 6x_2 \leq 0$$
$$-x_1 + 2x_2 \geq -2$$
$$-3x_1 - 3x_2 \geq -24$$
$$x_1 \geq 2$$
$$x_2 \geq 0$$

(a) Minimise $10x_1 - 4x_2$ (b) Maximise $10x_1 - 4x_2$ (c) Maximise $4x_1 + 10x_2$, (d) Minimise $4x_1 + 10x_2$

38. Solve graphically:

Minimise $Z = 12x_1 + 3x_2$

Subject to

$$4x_1 + 6x_2 \geq 24{,}000$$
$$x_1 + x_2 \geq 5{,}000$$
$$8x_1 + 2x_2 \geq 16{,}000$$
$$x_1, x_2 \geq 0$$

39. Maximise $3x_1 + 4x_2$ subject to the following constraints:

$$2x_1 + x_2 \leq 10;\ x_1 + 4x_2 \leq 36;\ x_1 + 2x_2 \leq 10;\ x_1 \geq 5;\text{ and } x_2 \geq 7$$

40. Find the maximum and the minimum values of the function $Z = 8x_1 + 5x_2$, subject to the following:

$$3x_1 - 2x_2 \geq 6$$
$$-2x_1 + 7x_2 \geq 7$$
$$2x_1 - 3x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

41. A local business firm is planning to advertise a special sale on radio and television during a particular week. A maximum budget of Rs 16,000 is approved for this purpose. It is found that radio commercials cost Rs 800 per 30-second spot with a minimum contract of five spots. Television commercials, on the other hand, cost Rs 4,000 per spot. Because of heavy demand only 4 television spots are still available in the week. Also it is believed that a TV spot is six times as effective as a radio spot in reaching consumers. How should the firm allocate its advertising budget to attract the largest number of consumers? How will the optimal solution be affected if the availability of TV spot is not constrained?

42. A company makes two products *X* and *Y*. Product *X* has a contribution of Rs 124 per unit and Product *Y* Rs 80 per unit.

Both products pass through two departments for processing and the times in minutes per unit are:

	Product *X*	Product *Y*
Department 1	150	90
Department 2	100	120

Currently, there is a maximum of 225 hours per week available in department 1 and 200 hours in department 2. The company can sell all it can produce of *X* but EEC quotas restrict the sale of *Y* to a maximum of 75 units per week.

The company, which wishes to maximise contribution margin, currently makes and sells 30 units of *X* and 75 units of *Y* per week.

The company is considering several possibilities including

(i) altering the production plan if it could be proved that there is a better plan than the current one;

(ii) increasing the availability of either department 1 or department 2 hours. The extra costs involved in increasing capacity are Rs 0.5 per hour for each department;

(iii) transferring some of their allowed sale quota for Product *Y* to another company. Because of commitments, the company would always retain a minimum sales level of 30 units.

You are required to

(a) calculate the optimum production plan using the existing capacities and state the extra contribution that would be achieved compared with the existing plan.

(b) advise the management whether they should increase the capacity of either department 1 or department 2 and, if so, by how many hours and what the resulting increase in contribution would be over that calculated in the improved production plan.

(c) calculate the minimum price per unit for which they could sell the rights to their own quota, down to the minimum level, given the plan in (a) as a starting point.

(ICMA, May, 1988, Adapted)

43. G Limited, manufacturer of superior garden ornaments, is preparing its production budget for the coming period. The company makes four types of ornaments, the data for which are as follows:

Product	Pixie	Elf	Queen	King
		(All values in Rs per unit)		
Direct Materials	25	35	22	25
Variable Overhead	17	18	15	16
Selling Price	111	98	122	326
Direct labour hours		(In Hours per unit)		
Type 1	8	6	–	–
Type 2	–	–	10	10
Type 3	–	–	5	25

Fixed overhead amounts to Rs 15,000 per period. Each type of labour is paid Rs 5 per hour but because of the skills involved, an employee of one type cannot be used for work normally done by another type. The maximum hours available each type are:

Type 1	8,000 Hours
Type 2	20,000 Hours
Type 3	25,000 Hours

The marketing department judges that, at the present selling prices, the demand for the products is likely to be:

Pixie	Unlimited Demand
Elf	Unlimited Demand
Queen	1,500 units
King	1,000 units

You are required

(a) to calculate the product mix that will maximise the profit, and the amount of the profit;

(b) to determine whether it would be worth while paying Type 1 Labour for overtime working at time and a half and, if so, to calculate the extra profit for each 1,000 hours of overtime;

(c) to comment on the principles used to find the optimum product mix in part (a), pointing out any possible limitations;

(d) to explain how a computer could assist in providing a solution for the data shown above.

(ICMA, November, 1993, Adapted)

44. The Hell Laboratories has a long history of production troubles. It produces two items, *A* and *B*, which are equally profitable. Recently, the company has entered into contract to supply 40 units of *A* and 20 units of *B* per week to another company. The technology of the chemical process implies that production of *A* must always be at least as large as of *B*. There are two raw material constraints to be satisfied:

$$5A + 8B \leq 400 \quad \text{and} \quad 55A + 50B \leq 2750.$$

Attempt the problem graphically and comment on the solution that you might obtain.

Chapter 3

Linear Programming II: Simplex Method

Chapter Overview

Once a linear programming problem is formulated, the next issue is to solve it for optimal values. The graphic method is indeed available but, as we know, it has its limitations. When only two products are produced by a firm, two investment opportunities are available to an investor, two advertising media are available to an advertising manager—in other words when only two decision variables are involved—that we can think of a graphic solution to an LPP. In case of more than two variables, the manager has to depend on what is called the Simplex method. Of course, this is not only the reason for using Simplex method. It is also a very powerful tool for solving linear programming problems. The Simplex and its variants can handle any complexities in the LPPs.

Besides solution to a problem, the method yields information on any unutilized resources—on whether the solution obtained to the given problem is the only optimal solution or are there equally attractive alternate solutions, on whether some given restriction is of no consequence (for example, raw materials may be available in plenty in a given situation), and so on. The method can also indicate if a given problem has no solution. While the solution to relatively small or moderately large problems may be found while working manually with the method, large scale problems can be solved with the help of computers where software are available. Real life problems obviously require computer help for solution.

Knowledge of simple algebraic manipulations and a good hand at arithmetic calculations are the pre-requisites for applying Simplex algorithm. A command over arithmetic operations on fractional values is necessary and desirable for Simplex calculations. Unfortunately, an ordinary electronic calculator is not of much help in this case.

Chapter 3

Linear Programming II: Simplex Method

3.1 INTRODUCTION

In the previous chapter we considered the formulation of linear programming problems and the graphic method of solving them. It was observed in the graphical approach to the solution of such problems that, in a given situation, the feasible region is determined by the set of constraints given in the problem while the objective function locates the optimal point—the one that *maximises* or *minimises*, as the case may be. Although this method is very efficient in developing the conceptual framework necessary for fully understanding the linear programming process, it suffers from the great limitation that it can handle problems involving only two decision variables. In the real-world situations, we frequently encounter cases where more than two variables are involved and, therefore, look for a method that can handle them. The *Simplex Method* provides an efficient technique which can be applied for solving LPPs of any magnitude—involving two or more decision variables. In this technique, the objective function is used to control the development and evaluation of each feasible solution to the problem.

The present chapter is devoted to a discussion of the simplex algorithm and demonstrates its application in solving linear programming problems.

3.2 SIMPLEX METHOD

The *'simplex algorithm'* is an iterative procedure for finding, in a systematic manner, the optimal solution to a linear programming problem. We have seen earlier that if a feasible solution to the problem exists, it is located at a corner point of the feasible region determined by the constraints of the system. The simplex method, according to its iterative search, selects this optimal solution from among the set of feasible solutions to the problem. The algorithm is indeed very efficient, because it considers only those feasible solutions which are provided by the corner points, and that too *not* all of them. Thus, by using this technique, we have to consider a minimum number of feasible solutions to obtain an optimal one. Also, this technique has the merit to indicate whether a given solution is optimal or not.

For applying simplex method to the solution of an LPP, first of all, an appropriately selected set of variables is introduced into the problem. The iterative process begins by assigning values only to these variables and the primary (decision) variables of the problem are all set equal to zero. This assumption is analogous to starting the evaluation process in the graphic approach at the point of origin, where both x_1 and x_2 are equal to zero. The algorithm then replaces one of the initial variables by another variable—the variable which contributes most to the desired optimal value enters in, while the variable creating the bottleneck to the optimal solution goes out. This improves the value of the objective function. This procedure of substitution of variables is repeated until no further improvement in the objective function value is possible. The algorithm terminates there indicating that the optimal solution is reached, or that the given problem has no solution.

3.2.1 Conditions for Application of Simplex Method

In order that the simplex method may be applied to a linear programming problem, the following two conditions have to be satisfied.

(1) The R.H.S. of each of the constraints, b_i should be non-negative. If an LPP has a constraint for which a negative resource value is given, it should be, in the first step, converted into positive value by multiplying both sides of the constraint by -1. For example, if the given constraint is $8x_1 - 3x_2 \geq -6$, it shall change into $-8x_1 + 3x_2 \leq 6$. Notice that the direction of inequality changes in the process.

(2) Each of the decision variables of the problem should be non-negative. Sometimes, a problem might have a variable which is 'unrestricted in sign' or 'free', so that it can assume negative values as well as non-negative. This situation is handled by treating such a variable as the difference of two variables which are both non-negative, because such a difference may be positive, negative or zero. Consider the following example:

Maximise $\quad Z = 5x_1 + 7x_2 - 2x_3$

Subject to

$$2x_1 + 5x_2 + 3x_3 \leq 80$$
$$5x_1 + 2x_2 - 2x_3 \leq 30$$
$$x_2 + 6x_3 \leq 42$$
$$x_1, x_2 \geq 0, x_3 \text{ unrestricted in sign}$$

Since x_3 is unrestricted in sign, we can represent it by a difference of two non-negative variables, say x_4 and x_5. Substituting $x_3 = x_4 - x_5$ in the above and simplifying, we get

Maximise $\quad Z = 5x_1 + 7x_2 - 2x_4 + 2x_5$

Subject to

$$2x_1 + 5x_2 + 3x_4 - 3x_5 \leq 80$$
$$5x_1 + 2x_2 - 2x_4 + 2x_5 \leq 30$$

$$x_2 + 6x_4 - 6x_5 \leq 42$$
$$x_1, x_2, x_4, x_5 \geq 0$$

After the solution is obtained, we shall substitute the difference of the values of x_4 and x_5 as the value of x_3.

The working of the *simplex method* proceeds by preparing a series of tables called *simplex tableaus*. We shall now discuss the application of the method—first for the *maximisation* and then for the *minimisation* case. Figure 3.2 gives the schematic of the simplex method.

3.3 SOLUTION TO MAXIMISATION PROBLEMS

Consider the linear programming problem given in Example 2.1, reproduced below.

Example 3.1 Maximise $Z = 40x_1 + 35x_2$ Profit

Subject to

$2x_1 + 3x_2 \leq 60$ Raw Material Constraint

$4x_1 + 3x_2 \leq 96$ Labour Hours Constraint

$x_1, x_2 \geq 0$

The solution to this problem is illustrated below in a step-wise manner.

Standardisation of the Problem

The first step in applying simplex method is to standardise the problem. For this, inequalities of the constraints are converted into equations. The first inequality of the system is

$$2x_1 + 3x_2 \leq 60$$

To convert it into an equation, we add a variable S_1 on the left hand side to get

$$2x_1 + 3x_2 + S_1 = 60$$

Now, if $2x_1 + 3x_2 = 60$, then $S_1 = 0$, and if $2x_1 + 3x_2 < 60$, then S_1 is equal to the difference. Therefore, S_1 can vary from 0 to 60 ($0 \leq S_1 \leq 60$) depending upon the value of $2x_1 + 3x_2$. The variable S_1 is referred to as a *slack variable* because it takes up any slack between the left and the right hand side of the inequality upon being converted into equation. In the context of the present problem, the value of S_1 shall indicate the amount of unused raw materials.

In a similar way, we introduce slack variable S_2 for converting the other inequality into equation, to get $4x_1 + 3x_2 + S_2 = 96$. Hence S_2 can vary between 0 and 96 ($0 \leq S_2 \leq 96$), depending upon the value of $4x_1 + 3x_2$, and it represents the labour hours not used.

We also need to modify the objective function here because the objective function for the simplex method should contain every variable in the system including slack or other variables (discussed later) added. We shall have to add S_1 and S_2 in the present case. But for this purpose, their coefficients also have to be determined. The coefficient may represent the cost involved in not using the raw materials or labour hours. We shall presume that the unused resources have no cost and, as such, they do not affect the profits. Thus, the coefficients assigned will be zeros.

The problem can now be expressed as follows:

Maximise $Z = 40x_1 + 35x_2 + 0S_1 + 0S_2$

Subject to

$$2x_1 + 3x_2 + S_1 + 0S_2 = 60$$

$$4x_1 + 3x_2 + 0S_1 + S_2 = 96$$

$$x_1, x_2, S_1, S_2 \geq 0$$

This is the standardised form of the given problem.

Obtaining the Initial Tableau and Solution

Using the standardised form, the information in the problem is presented in a tableau. The initial simplex tableau is given in Table 3.1.

Table 3.1 *Simplex Tableau*

Coefficient values from constraint equations

Basis		x_1	x_2	S_1	S_2	b_i
S_1	0	2	3	1	0	60
S_2	0	4	3	0	1	96
	c_j	40	35	0	0	
	Solution	0	0	60	96	
	$\Delta_j = c_j - z_j$	40	35	0	0	

Basis variables with their coefficients in the objective function

Constraint values

Contribution per unit

Values of basic and non-basic variables

To set up the tableau, we first list horizontally all the variables contained in the problem. Here, there are four variables: x_1, x_2, S_1 and S_2. Next, the coefficients in the constraint equations are written listing vertically the coefficients under their respective variables. It may be noted that each of the slack variables appears only in one equation. Therefore, the coefficient of each of the slack variables is taken to be zero in all the equations except the one in which it appears. Thus, in the first constraint, $2x_1 + 3x_2 + S_1 + 0S_2$, the co-efficients of x_1, x_2, S_1 and S_2 are 2, 3, 1 and 0 respectively. After putting the coefficients, the constraint values are mentioned on the right hand side against the rows. Finally, the row titled c_j indicates the coefficients of the various variables in the objective function, mentioned respectively in the various columns representing the variables. Having set up the simplex tableau, the next step is to locate the *identity* (matrix) and the variables involved in it. The identity contains all zeros except a diagonal column of positive 1's. The identity must have this square form with all zeros and a diagonal of (plus) ones. The size of this square would be determined by the number of constraints in the system. Locating the identity is of prime importance as the solution is identified in reference to this. To determine the solution, representing the first feasible solution, we set all variables other than those in the identity, equal to zero and then assign the values of the constants (b_i's) to variables in the identity. The variables in the identity are called *basic variables* and the remaining ones are called *non-basic variables.* In general, if a linear programming model has n variables and m constraints, then m variables would be basic variables and $n - m$ variables would be non-basic*. The basic variables form the *basis* and are known as the *variables in the solution.*

For our problem, the identity is formed by variables S_1 and S_2 and, therefore, they constitute the basis. Here S_1 and S_2 are basic and x_1 and x_2 are non-basic. The variables x_1 and x_2 are assigned zero values while S_1 and S_2 equal 60 and 96 respectively. For each row, the constraint value, b_i, is written against the variable in the identity having a 1 in that row. All the values are stated in a row in the simplex tableau, entitled *solution*, below the c_j row. Also, the basis is mentioned on the left hand side of the tableau, indicating the solution variables and their coefficients in the objective function. The remaining item to complete the first tableau is the bottom row containing the Δ_j values, whose calculation is explained in the next step involving the testing of optimality of the solution.

From the initial tableau, we observe the initial solution as $x_1 = 0, x_2, = 0, S_1 = 60$ and $S_2 = 96$. Substituting these values in the objective function, we get

$$Z = 40 \times 0 + 35 \times 0 + 0 \times 60 + 0 \times 96 = 0.$$

Thus, the initial feasible solution is not to produce any of the products A and B, with a zero profit.

A Note on the Solution The equations involving the constraints of the linear programming problem under consideration are reproduced below:

$$2x_1 + 3x_2 + S_1 = 60$$
$$4x_1 + 3x_2 + S_2 = 96$$

A careful look at these shows that there are four variables whose values are to be obtained while the equations are only two. It requires setting two variables equal to zero and then solving the equations for the other variables. In general, for a system $Ax = b$ of m linear equations in n variables, such that $n \geq m$, we need to set $n - m$ variables equal to zero and solve for the remaining m variables. Such a solution is called a *basic solution*. Obtaining a basic solution assumes that setting of the $n - m$ variables equal to zero each yields unique values for the remaining m variables, which implies that the columns for the remaining m variables are linearly independent. The set of $n - m$ variables is called *non-basic* variables and the m variables are termed the *basic variables*. Further, it may be noted that from n variables, a set of $n - m$ non-basic variables, or equivalently, m basic variables can be chosen in ${}^nC_m$ ways.

In the context of linear programming problems, when a problem has n variables and m constraints, we can have ${}^nC_m$ basic solutions. But since these problems require the variables to be non-negative, some of the basic solutions may not be feasible as they may involve negative values for some of the variables. Accordingly, a basic solution to $Ax = b$ in which all the variables are non-negative is termed as *basic feasible solution*. The intial solution for the problem under consideration is $x_1 = 0, x_2 = 0, S_1 = 60, S_2 = 96$, which indeed is a basic feasible solution.

The significance of basic feasible solution (bfs) stems from the fact that for any LPP, there is a unique extreme point of the feasible region corresponding to each *bfs* and also there is at least one *bfs* corresponding to each

* Under certain conditions some basic variables may also have zero values. This condition is called *degeneracy* and is discussed later.

extreme point of the feasible region. To illustrate, for the example under consideration, the graphic representation of constraints are reproduced in Figure 3.1 and the constraints in the standardised form are presented below:

$$2x_1 + 3x_2 + S_1 + 0S_2 = 60$$
$$4x_1 + 3x_2 + 0S_1 + S_2 = 96$$

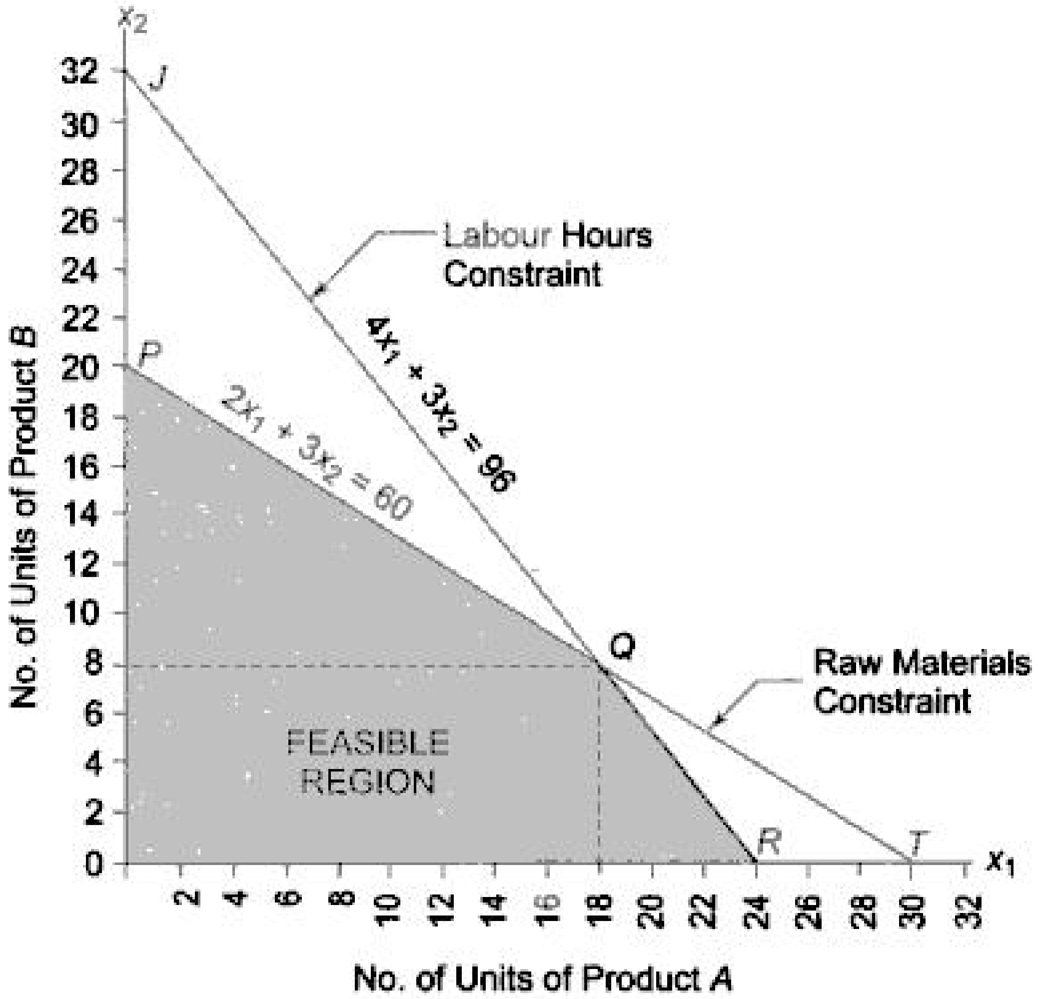


Fig. 3.1 Graphic Presentation of Constraints

With 4 variables and 2 equations, there shall be ${}^4C_2 = 6$ basic solutions are detailed as follows:

Non-basic variables	*Basic variables*	*Solution*	*bfs?*	*Corner point*
S_1, S_2	x_1, x_2	$S_1 = 0,\ S_2 = 0,\ x_1 = 18,\ x_2 = 8$	Yes	Q
x_2, S_2	x_1, S_1	$x_2 = 0,\ S_2 = 0,\ x_1 = 24,\ S_1 = 12$	Yes	R
x_2, S_1	x_1, S_2	$x_2 = 0,\ S_1 = 0,\ x_1 = 30,\ S_2 = -24$	No	–
x_1, S_2	x_2, S_1	$x_1 = 0,\ S_2 = 0,\ x_2 = 32,\ S_1 = -36$	No	–
x_1, S_1	x_2, S_2	$x_1 = 0,\ S_1 = 0,\ x_2 = 20,\ S_2 = 36$	Yes	P
x_1, x_2	S_1, S_2	$x_1 = 0,\ x_2 = 0,\ S_1 = 60,\ S_2 = 96$	Yes	0

Observe here that two of the solutions are not basic feasible solutions since they involve negative value for a variable, and are not relevant, while each of the remaining are basic feasible solutions and they correspond to an extreme point on the graph of the feasible region.

Thus, extreme points of the feasible region of an LPP represent the basic feasible solutions to the problem and in looking for the optimal solution to an LPP, we need only locate the best basic feasible solution. Further, each simplex tableau represents a basic feasible solution except only when there is an 'artificial' variable in the basis (a situation discussed in Example 3.3).

Testing the Optimality

To test whether the solution obtained in simplex tableau 1 is optimal or not, we calculate Δ_j values as being equal to $c_j - z_j$, which are given at the bottom row of Table 3.1. To obtain the value of z_j under each variable head column, first each element of that column is multiplied by the corresponding coefficient of the solution variables appearing in the basis. Then the products are added up and we get z_j. The values of z_j represent the amount by which the profit would be reduced if one unit of any of the variables (x_1, x_2, S_1, or S_2) were added to the mix. For instance, to introduce one unit of x_1, the entries in this column indicate that 2 kg of raw material and 4 labour hours must be given up. Since the unused resources are assumed to have zero cost, there would be no reduction in the profit. Hence, $z_1 = 0 \times 2 + 0 \times 4 = 0$.

With c_j as the profit per unit, $c_1 = 40$, $\Delta_j = c_j - z_j$ represents the net profit which would result from introducing one unit of variable to the product mix–that is, the solution. Since a unit of x_1 adds Rs 40 to the profit and its introduction causes no loss, $\Delta_1 = 40 - 0 = 40$.

Now from the tableau, for column headed	x_1,	$z_1 = 0 \times 2 + 0 \times 4 = 0$	$\therefore \Delta_1 = 40 - 0 = 40$;
" " " " " "	x_2,	$z_2 = 0 \times 3 + 0 \times 3 = 0$	$\therefore \Delta_2 = 35 - 0 = 35$;
" " " " " "	S_1,	$z_3 = 0 \times 1 + 0 \times 0 = 0$	$\therefore \Delta_3 = 0 - 0 = 0$;
" " " " " "		$z_4 = 0 \times 0 + 0 \times 1 = 0$	$\therefore \Delta_4 = 0 - 0 = 0$.

The Δ_j row is also called the *net-after-opportunity-cost row*, or the net evaluation row (NER).

The Test As indicated earlier, except when an *artificial* variable is included in the basis, a simplex tableau depicts an optimal solution if all entries in the Δ_j row are

(a) Zero or negative (*i.e.* all $\Delta_j \le 0$) when the LPP is of maximisation type, and
(b) Zero or positive (*i.e.* all $\Delta_j \ge 0$) when the LPP is of minimisation type.

Now, looking at the Δ_j values in simplex tableau 1, we observe that the solution is not optimal because it contains positive values.

Deriving a Revised Tableau for Improved Solution

The presence of positive Δ_j values suggests that the solution can be improved upon by moving any one of the variables into the solution that are not there.

Each of the values in the Δ_j row signifies the amount of increase in the objective function that would occur if one unit of the variable (represented by the column head) were introduced into the solution. We select the variable that has the largest Δ_j value—the variable x_1. This is designated as the *incoming variable* and the selection of it is indicated by an arrow under the column headed x_1 in the Table 3.2 which reproduces the information in Table 3.1.

Table 3.2 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	2	3	1	0	60	30
S_2	0	4*	3	0	1	96	24 ← Outgoing variable (key row)
c_j		40	35	0	0		
Solution		0	0	60	96		
Δ_j		40 ↑ Incoming variable (key column)	35	0	0		

The column corresponding to this variable is called the *key column*. Next, the b_i values are divided by the corresponding values in the key column, and we get the ratios b_i/a_{ij}, also called the *replacement ratios*. The row with the least non-negative quotient is then selected. It is called the *key row* and the variable corresponding to this represents the *outgoing variable*. The element which lies at the intersection of the key column and key row is termed as the *key element*. In the Table 3.2, we observe that the key row is represented by the arrow mark—the row with outgoing variable S_2. The element is 4, with the asterisk mark (*).

Using the information on key column, key row and key element, another tableau is derived wherein the various elements are obtained as given here.

(a) Divide each element of the key row (including b_i) by the key element to get the corresponding values in the new tableau. The row of values so derived is called the *replacement row*.
For our example, the replacement row would be:
1 3/4 0 1/4 24

(b) For each row other than the key row,

$$\text{New row element} = \text{Old row element} - \left(\begin{matrix}\text{Row element in the} \\ \text{key column}\end{matrix} \times \begin{matrix}\text{Corresponding replacement} \\ \text{row value}\end{matrix}\right)$$

Accordingly, for the other row, the values shall be:

Old row element	*Minus*	*Row element in key column*	×	*Corresponding replacement row value*	=	*New element*
2	–	2	×	1	=	0
3	–	2	×	3/4	=	3/2
1	–	2	×	0	=	1
0	–	2	×	1/4	=	–1/2
60	–	2	×	24	=	12

The revised simplex tableau is given in Table 3.3.

Table 3.3 *Simplex Tableau 2: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	0	3/2*	1	–1/2	12	8 ← Outgoing variable
x_1	40	1	3/4	0	1/4	24	32
c_j		40	35	0	0		
Solution		24	0	12	0		
Δ_j		0	5 ↑ Incoming variable	0	–10		

In this tableau, the c_j and the 'solution' values are written in the same way as in the previous tableau. According to this, the solution to the problem is: $x_1 = 24$, $x_2 = 0$, $S_1 = 12$ and $S_2 = 0$. Substituting these values in the objective function, we get

$$Z = 40 \times 24 + 35 \times 0 + 0 \times 12 + 0 \times 0 = 960.$$

The solution calls for producing 24 units of product *A* and none of the product *B*. A total of 12 kg of raw material would remain unutilised while the labour hours would be fully used by this decision which would bring in a profit of Rs 960. Notice that this decision corresponds to the corner point *R* in the graph in the Figure 3.1.

Now we shall test whether this solution is optimal or not. For this purpose, the Δ_j values are derived in a similar manner as before. Here they all are not less than, or equal to, zero. Therefore, the solution is not optimal.

Revised Tableau for Further Improved Solution

To obtain an improved solution to the problem, we proceed to derive a revised tableau as follows.

(a) Obtain the key column, key row and key element. The largest Δ_j, equal to 5, corresponds to the variable x_2. Thus, it is taken to be the incoming variable and the column is marked as *key column*. Using the a_{ij} values of the key column, the replacement ratios are found to be 8 and 32. Accordingly, the first row is the *key row* and the outgoing variable is S_1.

The intersection of the key column and key row yields the key element as 3/2, which is shown with an asterisk mark in Table 3.3.

(b) With information about key column, key row and key element, we can now obtain a revised simplex tableau in a manner similar to the one discussed earlier. This is contained in Table 3.4.

Table 3.4 *Simplex Tableau 3: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	b_i
x_2	35	0	1	2/3	–1/3	8
x_1	40	1	0	–1/2	1/2	18
c_j		40	35	0	0	
Solution		18	8	0	0	
Δ_j		0	0	–10/3	–25/3	

The solution given by this corresponds to another extreme point *Q*, of the Figure 3.1. This solution, $x_1 = 18$, $x_2 = 8$, yields the objective function value $40 \times 18 + 35 \times 8 + 0 \times 0 + 0 \times 0 = 1{,}000$. By this, both the resources would be fully utilized. The values in the Δ_j row, being ≤ 0, indicate that the solution is optimal.

Besides indicating optimality, the Δ_j row values in the final tableau give significant information—the shadow prices of the resources. This is discussed in detail in the next chapter.

3.3.1 Justification and Significance of Elements in Simplex Tableau

We examine now the meaning and significance of the b_i values, the substitution rates (contained in the body of simplex tableau) and the Δ_j values in turn, first for Simplex Tableau 2 and then for Simplex Tableau 3.

For Simplex Tableau 2 The tableau is reproduced in Table 3.5

To conclude, then, a positive number in the Δ_j row indicates that the objective function value may be increased further while a negative element in the row reflects the loss (in profit) that would result if one unit of the variable heading the particular variable were added to the solution.

For Simplex Tableau 3 This is reproduced in Table 3.6 and an explanation of its elements follows:

Table 3.6 *Simplex Tableau 3*

Basis		x_1	x_2	S_1	S_2	b_i
x_2	35	0	1	2/3	–1/3	8
x_1	40	1	0	–1/2	1/2	18
c_j		40	35	0	0	
Solution		18	8	0	0	
Δ_j		0	0	–10/3	–25/3	

(i) The b_i values indicate the output of products *B* and *A* to be 8 and 18 units respectively.

(ii) The substitution rates and the Δ_j values under different variables indicate the following:

The values 0 and 1 under x_1 suggest that a unit of x_1 added to the current product-mix will call for replacing one unit of the same product in the solution with no change needed in the other product. Thus, addition to profit for adding a unit of product *A* (i.e. x_1) = Rs 40, and loss of profit for replacing a unit from the existing mix is also equal to Rs 40. Accordingly, the net change in profit, $\Delta_j = 40 - 40 = 0$.

A similar interpretation may be given for the values 1 and 0, and $\Delta_j = 0$ in respect of the variable x_2.

The substitution rates under S_1 are 2/3 against x_2 and –1/2 against x_1. These imply that releasing one kg of raw material will entail losing 2/3 units of product *B* and gaining 1/2 unit of product *A* (since one unit of product *B* requires 3 kg of raw material, reducing 2/3 unit of it will release $3 \times 2/3 = 2$ kg while adding 1/2 unit of x_1 would need one kg of raw material since one unit of product *A* requires 2 kg of raw material).

The loss of 2/3 unit of x_2 and the gain of 1/2 unit of x_1 would result in a net loss of $(35 \times 2/3) - (40 \times 1/2) =$ Rs 10/3. This is indicated by $\Delta_j = -10/3$.

In a similar way, it may be observed that the values –1/3 and 1/2 under S_2 indicate the substitution rates with respect to x_2 and x_1 respectively. Withdrawing an hour of labour would cause a reduction of 1/2 unit of product *A* and addition of 1/3 unit of product *B*. The reduction of 1/2 unit of *A* would release $4 \times 1/2 = 2$ hours of labour while addition of 1/3 unit of *B* would consume $3 \times 1/3 = 1$ hour, resulting in a net release of one hour of labour as desired. These changes shall cause a net reduction of Rs 25/3 in profit – a loss of $40 \times 1/2$ = Rs 20 and gain of $35 \times 1/3$ = Rs 35/3 – as indicated by the Δ_j value.

Example 3.2 A firm produces three products *A*, *B*, and *C*, each of which passes through three departments: Fabrication, Finishing and Packaging. Each unit of product *A* requires 3, 4 and 2; a unit of product *B*

Following the approach already discussed, we first introduce some new variables to convert inequalities of the system into equations. The variable required for converting a 'greater than' type of inequality into an equation is called *surplus variable* and it represents the excess of what is generated (given by the LHS of the inequality) over the requirement (shown by the RHS value b_i). With surplus variables, S_1 and S_2 respectively for the first and the second constraints, the augmented problem shall be

Minimise $Z = 40x_1 + 24x_2 + 0S_1 + 0S_2$

Subject to

$$20x_1 + 50x_2 - S_1 = 4{,}800$$
$$80x_1 + 50x_2 - S_2 = 7{,}200$$
$$x_1, x_2, S_1, S_2 \geq 0$$

Now, as soon as we proceed to the next step we experience a problem which is like this. We know that the simplex method needs an initial solution to get the process started. In this case, it is easy to visualize that an initial solution does not exist because, if we let x_1 and x_2 each equal to zero, we get $S_1 = -4{,}800$ and $S_2 = -7{,}200$, which is not feasible as it violates the *non-negativity* restriction. In terms of the simplex tableau, when we write all the information, we observe that we do not get identity because unlike in case of slack variables, the coefficient values of surplus variables S_1 and S_2 appear as minus one (–1).

3.5 BIG-M METHOD

In a case where an identity is not obtained, as in the problem under consideration, a variant of the simplex method called the Big-M method is employed. In this method, we add *artificial variables* into the model to obtain an initial solution. Unlike *slack* or *surplus* variables, artificial variables have no tangible relationship with the decision problem. Their sole purpose is to provide an initial solution to the given problem.

When artificial variables are introduced in the problem under consideration, its constraints appear as

$$20x_1 + 50x_2 - S_1 + A_1 = 4{,}800$$
$$80x_1 + 50x_2 - S_2 + A_2 = 7{,}200$$

It is significant to understand that the artificial variables, which are not seen to disturb the equations already obtained since they are not 'real', are introduced for the limited purpose of obtaining an initial solution and are required for the constraints of '≥' type, or the constraints with '=' sign. It is not relevant whether the objective function is of the minimisation or the maximisation type. Obviously, since artificial variables do not represent any quantity relating to the decision problem, they must be driven out of the system and must not show in the final solution (and if at all they do, it represents a situation of infeasibility, which is discussed later in this chapter). This can be ensured by assigning an extremely high cost to them. Generally, a value M is assigned to each artificial variable, where M represents a number higher than any finite number. For this reason, the method of solving the problems where artificial variables are involved is termed as the ***Big-M Method***. When the problem is of the minimisation nature, we assign in the objective function a coefficient of $+M$ to each of the artificial variables. On the other hand, for the problems with the objective function of maximisation type, each of the artificial variables introduced has a coefficient $-M$. Note that it is attempted to prohibit the appearance of artificial variables in the solution by assigning these coefficients: an extremely large value when the objective is to minimise and an extremely small (negative) value when it is desired to maximise the objective function.

For our present example, the objective function would appear as

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

The optimal solution to the problem is: $x_1 = 2$ and $x_2 = 12$, $S_1 = 2$ and other variables = 0. The objective function value is $2 \times 2 + 4 \times 12 = 52$.

3.6 TWO-PHASE METHOD

As an alternative to the *Big-M* method, there is also available another method for dealing with linear programming problems involving artificial variables. This is called the *two-phase method* and, as its name implies, it separates the solution procedure into two phases. In phase-I, all the artificial variables are eliminated from the basis. If a feasible solution is obtained in this phase, which has no artificial variables in the basis in the final tableau, then we proceed to phase II. In this phase, we use the solution from phase I as the *initial* basic feasible solution and use the simplex method to determine the optimal solution. The method is discussed in the following paragraphs.

Suppose that we are given the following LPP:

Minimise $$Z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad j = 1, 2, \ldots, n$$

To solve it, we follow steps detailed here.

3.6.1 Phase I

Step 1 By subtracting a surplus variable, convert each of the constraints into equality relationships and then add an artificial variable.

Step 2 Assign zero coefficients to each of the primary (x_j) variables and to the surplus variables; and assign unit coefficients to each of the artificial variables (in a maximisation problem, the coefficients to each of the artificial variables shall be –1). This amounts to replacing the objective function of the original problem by the sum of the artificial variables.

The assignment of the stated coefficients yields the following auxiliary problem:

Minimise $$Y = \sum_{j=1}^{n} 0x_j + \sum_{i=1}^{m} 0S_i + \sum_{i=1}^{m} 1A_i \text{ or Min. } \Sigma A_i$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j - S_i + A_i = b_i \quad i = 1, 2, \ldots, m$$

and $$x_j, S_i, A_i \geq 0 \text{ for all } i \text{ and } j$$

Table 3.21 *Simplex Tableau 5: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	b_i
x_2	24	8/5	1	0	– 1/50	144
S_1	0	60	0	1	– 1	2400
c_j		40	24	0	0	
Solution		0	144	2,400	0	
Δ_j		8/5	0	0	12/25	

The optimal solution given by Simplex Tableau 5 is $x_1 = 0$ and $x_2 = 144$. It is the same as obtained earlier.

3.7 SOME SPECIAL TOPICS

3.7.1 Multiple Optimal Solutions

We have already seen in the previous chapter that the solution to a linear programming problem may or may not be unique. It was established in the graphic solution to the LPPs that when the iso-profit (or iso-cost) line has the same slope as that of a constraint, then the optimal solution to the given problem may not be unique and, instead, multiple optima may exist.

To demonstrate the existence of multiple optimal solutions using the simplex method, we consider again the data of Example 2.4 reproduced here.

Example 3.6 Maximise $Z = 8x_1 + 16x_2$

Subject to

$$x_1 + x_2 \leq 200$$
$$x_2 \leq 125$$
$$3x_1 + 6x_2 \leq 900$$
$$x_1, x_2 \geq 0$$

Solution to this problem using simplex algorithm is contained in Tables 3.22 through 3.24.

Table 3.22 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	1	1	1	0	0	200	200
S_2	0	0	1*	0	1	0	125	125 ←
S_3	0	3	6	0	0	1	900	150
c_j		8	16	0	0	0		
Solution		0	0	200	125	900		
Δ_j		8	16	0	0	0		
			↑					

Table 3.29 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2'	S_1	S_2	S_3	A_1	b_i	b_i/a_{ij}
S_1	0	2	1	1	0	0	0	40	20
S_2	0	4*	−1	0	1	0	0	20	5 ←
A_1	−1	1	0	0	0	−1	1	30	30
c_j		1	0	0	0	0	−1		
Solution		0	0	4	20	0	30		
Δ_j		1	0	0	0	−1	0		
		↑							

Table 3.30 *Simplex Tableau 2: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	A_1	b_i	b_i/a_{ij}
S_1	0	0	3/2*	1	−1/2	0	0	30	20 ←
x_1	0	1	−1/4	0	1/4	0	0	5	−20
A_1	−1	0	1/4	0	−1/4	−1	1	25	100
c_j		0	0	0	0	0	−1		
Solution		5	0	30	0	0	25		
Δ_j		0	1/4	0	−1/4	−1	0		
			↑						

Table 3.31 *Simplex Tableau 3: Final, Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	A_1	b_i
x_2	0	0	1	2/3	−1/3	0	0	20
x_1	0	1	0	1/6	1/6	0	0	10
A_1	−1	0	0	−1/6	−1/6	−1	1	20
c_j		0	0	0	0	0	−1	
Solution		10	20	0	0	0	20	
Δ_j		0	0	−1/6	−1/6	−1	0	

3.7.3 Unboundedness

As discussed in Chapter 2, an LPP is said to have an unbounded solution if its objective function value can be increased (in case it is a maximisation problem) or decreased (if it is a minimisation problem) without limit. Now we consider as to how unboundedness may be discovered using the simplex method.

The initial solution is given in Table 3.36.

Table 3.36 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}	
S_1	0	6	3	1	0	0	18	6 ↑	
S_2	0	3	1	0	1	0	8	8	Tie
S_3	0	4	5	0	0	1	30	6 ↓	
c_j		28	30	0	0	0			
Solution		0	0	18	8	30			
Δ_j		28	30	0	0	0			
			↑						

In this tableau we observe that there is a tie between the first and the third rows and, therefore, either of the variables S_1 or S_3 could be taken to be the outgoing variable. We shall consider them one by one. By taking S_1 as the outgoing variable, the revised tableau may be drawn as shown in Table 3.37.

Table 3.37 *Simplex Tableau 2: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i
x_2	30	2	1	1/3	0	0	6
S_2	0	1	0	– 1/3	1	0	2
S_3	0	– 6	0	– 5/3	0	1	0
c_j		28	30	0	0	0	
Solution		0	6	0	2	0	
Δ_j		– 32	0	– 10	0	0	

This tableau gives the optimal solution $x_1 = 0$, and $x_2 = 6$, with objective function value equal to 180. Now, deleting S_3, the revised simplex tableau would appear as shown in Table 3.38.

Table 3.38 *Simplex Tableau 3: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	18/5*	0	1	0	– 3/5	0	0 ←
S_2	0	11/5	0	0	1	– 1/5	2	10/11
x_2	30	4/5	1	0	0	1/5	6	15/2
c_j		28	30	0	0	0		
Solution		0	6	0	2	0		
Δ_j		4	0	0	0	– 6		
		↑						

Table 3.42 *Simplex Tableau 3: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i
x_2	2	0	1	3/2	– 2	0	6
x_1	5	1	0	– 1/2	1	0	1
S_3	0	0	0	3	– 5	1	12
c_j		5	2	0	0	0	
Solution		1	6	0	0	12	
Δ_j		0	0	– 1/2	– 1	0	

This solution is optimal with $x_1 = 1$, $x_2 = 6$ and $Z = 17$. It is *non-degenerate.*

When S_3 is Eliminated

Beginning with the initial solution given in Table 3.40, if we select S_3 as the outgoing variable, the optimal solution may be reached as contained in Tables 3.43 through 3.45.

Table 3.43 *Simplex Tableau 4: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	10/3	1	0	– 4/3	4	6/5
S_2	0	0	2*	0	1	– 1	0	0 ←
x_1	5	1	– 1/3	0	0	1/3	3	– 1
c_j		5	2	0	0	0		
Solution		3	0	4	0	0		
Δ_j		0	11/3	0	0	– 5/3		
			↑					

For the solution in Table 3.43 with $x_1 = 3$ and $x_2 = 0$, we have $Z = 15$. The solution is degenerate and non-optimal. To improve the solution, we select the least non-negative replacement ratio equal to 0 for S_2, since the denominator here is positive. The improved solution is contained in Table 3.44.

Table 3.44 *Simplex Tableau 5: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	0	1	– 5/3	1/3*	4	12 ←
x_2	2	0	1	0	1/2	– 1/2	0	–
x_1	5	1	0	0	1/6	1/6	3	18
c_j		5	2	0	0	0		
Solution		3	0	4	0	0		
Δ_j		0	0	0	– 11/6	1/6		
						↑		

Thus, optimal solution to the problem is: $x_1 = 175/17$, $x_2 = x_3 - x_4 = (0 - 72/17) = -72/17$ and the objective function value is $(8 \times 175/17) - (4 \times (-72/17)) = 1688/17$.

Example 3.13 Maximise $Z = 6x_1 + 20x_2$

Subject to

$$2x_1 + x_2 \leq 32$$
$$3x_1 + 4x_2 \leq 80$$
$$x_1 \geq 8$$
$$x_2 \geq 10$$

Sometimes, in a given problem it may be provided that one (or more) of the variables may not assume a value higher and/or lower than a particular value. The problem may be solved conveniently if the variables are restructured. In the given problem, the variable x_1 is sought to be at least equal to 8 and x_2 no less than 10. Although this problem can be solved by using slack variables in each of the first two constraints, and surplus and artificial variables in the other two, we can reframe the problem in the following way in order to save computational time and effort. Since $x_1 \geq 8$, we have $x_1 - x_3 = 8$ (x_3 being the surplus variable) with $x_1 = 8 + x_3$ and $x_2 = 10 + x_4$ (determined similarly), we can rewrite the problem as follows:

Maximise $Z = 6(8 + x_3) + 20(10 + x_4)$ or $Z = 6x_3 + 20x_4 + 248$

Subject to

$$2(8 + x_3) + (10 + x_4) \leq 32 \quad \text{or} \quad 2x_3 + x_4 \leq 6$$
$$3(8 + x_3) + 4(10 + x_4) \leq 80 \quad \text{or} \quad 3x_3 + 4x_4 \leq 16$$
$$x_3, x_4 \geq 0$$

Introducing the necessary slack variables, we get

Maximise $Z = 6x_3 + 20x_4 + 248 + 0S_1 + 0S_2$

Subject to

$$2x_3 + x_4 + S_1 = 6$$
$$3x_3 + 4x_4 + S_2 = 16$$
$$x_3, x_4, S_1, S_2 \geq 0$$

The solution is contained in Tables 3.49 and 3.50.

Table 3.49 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_3	x_4	S_1	S_2	b_i	b_i/a_{ij}
S_1	0	2	1	1	0	6	6
S_2	0	3	4*	0	1	16	4 ←
c_j		6	20	0	0		
Solution		0	0	6	16		
Δ_j		6	20 ↑	0	0		

Table 3.56 *Simplex Tableau 3: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	0	1/2*	−1	0	−1/2	1	0	50	100 ←
x_1	2	1	1/2	0	0	1/2	0	0	300	600
S_2	0	0	1/2	0	1	1/3	0	−1	175	350
c_j		2	3	0	0	0	M	M		
Solution		300	0	0	175	0	50	0		
Δ_j		0	$2-\frac{M}{2}$ ↑	M	0	$\frac{M}{2}-1$	0	M		

Table 3.57 *Simplex Tableau 4: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	A_1	A_2	b_i
x_2	3	0	1	−2	0	−1	2	0	100
x_1	2	1	0	1	0	1	−1	0	250
S_2	0	0	0	1	1	1	−1	−1	125
c_j		2	3	0	0	0	M	M	
Solution		250	100	0	125	0	0	0	
Δ_j		0	0	4	0	1	$M-4$	M	

Thus, the optimal solution is: $x_1 = 250$ and $x_2 = 100$, with $Z = 800$. The solution is unique optimal solution and no multiple solutions exist, because none of the non-basic variables has Δ_j equal to zero.

Example 3.16 A firm uses three machines in the manufacture of three products. Each unit of product *A* requires 3 hours on machine I, 2 hours on machine II and one hour on machine III. Each unit of product *B* requires 4 hours on machine I, one hour on machine II and 3 hours on machine III, while each unit of product *C* requires 2 hours on each of the three machines. The contribution margin of the three products is Rs 30, Rs 40 and Rs 35 per unit respectively. The machine hours available on three machines are 90, 54 and 93 respectively.

(i) Formulate the above problem as a linear programming problem.
(ii) Obtain optimal solution to the problem by using the simplex method. Which of the three products shall not by produced by the firm? Why?
(iii) Calculate the percentage of capacity utilisation in the optimal solution.
(iv) What are the shadow prices of the machine hours?
(v) Is the optimal solution degenerate?

(i) Let x_1, x_2 and x_3 represent the output of products *A*, *B* and *C*, respectively. With the given contribution margins, resource requirements and availability, the LPP can be expressed as follows:

Maximise $\quad Z = 30x_1 + 40x_2 + 35x_3 \quad$ Contribution

Table 3.63 *Simplex Tableau 3: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	0	0	1	2	0	−1	−2	1	18,000	9,000
x_2	1/10	0	1	0	−1	0	1	1	−1	6,000	–
S_3	0	0	0	0	1*	1	−1	−1	1	6,000	6,000 ←
x_1	7/100	1	0	0	−1	0	0	1	0	6,000	–
c_j		7/100	1/10	0	0	0	0	$-M$	$-M$		
Solution		6,000	6,000	18,000	0	6,000	0	0	0	$Z = 1,020$	
Δ_j		0	0	0	$\frac{17}{100}$ ↑	0	−1/10	$-M-\frac{17}{100}$	$-M+\frac{1}{10}$		

Table 3.64 *Simplex Tableau 4: Non-optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	A_1	A_2	b_i	b_i/a_{ij}
S_1	0	0	0	1	0	−2	1*	0	−1	6,000	6000 ←
x_2	1/10	0	1	0	0	1	0	0	0	12,000	–
S_2	0	0	0	0	1	1	−1	−1	1	6,000	–
x_1	7/100	1	0	0	0	1	−1	0	1	12,000	–
c_j		7/100	1/10	0	0	0	0	$-M$	$-M$		
Solution		12,000	12,000	6,000	6,000	0	0	0	0	$Z = 2,040$	
Δ_j		0	0	0	0	$-\frac{17}{100}$	$\frac{7}{100}$ ↑	$-M$	$-M-\frac{7}{100}$		

Table 3.65 *Simplex Tableau 5: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	A_1	A_2	b_i
S_4	0	0	0	1	0	−2	1	0	−1	6,000
x_2	1/10	0	1	0	0	1	0	0	0	12,000
S_2	0	0	0	1	1	−1	0	−1	0	12,000
x_1	7/100	1	0	1	0	−1	0	0	0	18,000
c_j		7/100	1/10	0	0	0	0	$-M$	$-M$	
Solution		18,000	12,000	0	12,000	0	6,000	0	0	$Z = 2,460$
Δ_j		0	0	−7/100	0	−3/100	0	$-M$	$-M$	

Example 3.19 From the following initial simplex tableau:

Basis	c_j	x_1	x_2	S_1	S_2	A_1	A_2	
		15	25	0	0	$-M$	$-M$	
A_1	$-M$	7	6	-1	0	1	0	20
S_2	0	8	5	0	1	0	0	30
A_2	$-M$	3	-2	0	0	0	1	18
z_j		$-10M$	$-4M$	M	0	$-M$	$-M$	$-38M$
$c_j - z_j$		$15 + 10M$	$25 + 4M$	$-M$	0	0	0	

write down the original primal problem represented by the above tableau.

Find out the optimal solution of this problem. Is it a unique solution? Why? *(M Com, Delhi, 1996)*

The given information reveals that

(i) There are two decision variables x_1 and x_2, with objective function coefficients equal to 15 and 25, respectively.

(ii) There are three constraints with RHS values as 20, 30 and 18, and involving $\geq$, $\leq$ and $=$ signs, respectively. This is indicated by the slack, surplus and artificial variables.

(iii) The objective function is of maximisation type because the artificial variables bear negative coefficients ($= -M$) in the objective function.

The required LPP is, therefore,

Maximise $Z = 15x_1 + 25x_2$

Subject to

$$7x_1 + 6x_2 \geq 20$$
$$8x_1 + 5x_2 \leq 30$$
$$3x_1 - 2x_2 = 18$$
$$x_1, x_2 \geq 0$$

The problem is solved as follows. Introducing the necessary slack, surplus and artificial variables, we have,

Maximise $Z = 15x_1 + 25x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$$7x_1 + 6x_2 - S_1 + A_1 = 20$$
$$8x_1 + 5x_2 + S_2 = 30$$
$$3x_1 - 2x_2 + A_2 = 18$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

The solution is given in Tables 3.70 through 3.72.

29. If a certain solution to an LPP is degenerate and non-optimal, the next solution would necessarily be degenerate.
30. If the non-negative replacement ratios are tied, then multiple optimal solutions are indicated.

EXERCISES

1. Explain the concept and computational steps of the simplex method for solving linear programming problems. How would you identify whether an optimal solution to a problem obtained using simplex algorithm is unique or not?
2. (a) What is the difference between a feasible solution, a basic feasible solution, and an optimal solution of a linear programming problem?
   (b) What is the difference between simplex solution procedure for a 'maximisation' and a 'minimisation' problem?
   (c) Using the concept of net contribution, provide an intuitive explanation of why the criterion for optimality for maximisation problems is different from that of minimisation problems.
3. Outline the steps involved in the simplex algorithm for solving a linear programming maximisation problem. Also define the technical terms used therein.
4. What is the difference between slack, surplus, and artificial variables? How do they differ in their structure and use?
5. State the conditions required for applying simplex method to a linear programming problem. How do we proceed in a case when both of these are not met?
6. In solving a linear programming problem by simplex method, explain how you would move from a given basic feasible solution to another basic feasible solution with an improved value of the objective function.
7. How are key column and key row determined in a simplex tableau containing a non-optimal solution? How would you proceed if a tie is obtained in either of them?
8. Describe the two-phase method for solving the linear programming problems.
9. Consider each of the following statements and state whether it is true or false. In case it is false, write the correct statement.
   (i) Each inequality constraint in a linear programming problem adds only one variable, when it is solved by the simplex method.
   (ii) All the slack, surplus and artificial variables are included in the basis of the first simplex tableau.
   (iii) If optimal solution to an LPP is degenerate, it is of no consequence to the manager.
   (iv) Any solution which satisfies at least one of the constraints in an LPP is included in the feasible region of the problem.
   (v) For solving by simplex method, only an artificial variable is needed to be introduced if a constraint in an LPP involves an equation. Slack and surplus variables are not required in such a case.
10. Explain and graphically illustrate infeasibility and unboundedness. How can each of these be detected while applying simplex technique?
11. 'In a given problem, the value of objective function improves with successive iterations.' Is there any exception to this? Explain clearly.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

(b) Find out the product mix so as to maximise profit.

(c) Show that the total available hours of *X* and *Y* have been fully utilised and there is surplus hours of *Z*. How many hours are surplus in machine centre *Z*?

14. A pharmaceutical company has 100 kg of material *A*, 180 kg of material *B* and 120 kg of material *C* available per month. They can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage by weight of material *A*, material *B* and material *C* respectively, in each of the products and the balance represents inert ingredients. The cost of raw material is given below:

Ingredient	*Cost per kg (Rs)*
Material *A*	80
Material *B*	20
Material *C*	50
Inert ingredient	20

Selling price of these products is Rs 40.50, Rs 43 and Rs 45 per kg respectively. There is a capacity restriction of the company for the product 5-10-5, that is, they cannot produce more than 30 kg per month. Formulate a linear programming model for maximising the monthly profit.

Determine how much of each of the products should they produce in order to maximise their monthly profits. *(CS, June, 1992)*

15. Noah's Boats makes three different types of boats. All boats can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximises his revenue, in view of the information given in the following table:

Input	*Row boat*	*Canoe*	*Kayak*	*Monthly availability*
Labour (hours)	12	7	9	1,260 hours
Wood (Board feet)	22	18	16	19,008 board feet
Screws (kg)	2	4	3	396 kg
Selling price (in Rupees)	4,000	2,000	5,000	

(a) Formulate the above as a linear programming problem.

(b) Solve it by the simplex method.

From the optimal table of the solved linear programming problem, answer the following questions:

(c) How many boats of each type will be produced and what will be the resulting revenue?

(d) Which, if any, of the resources are not fully utilised? If so, how much of spare capacity is left?

(e) How much wood will be used to make all of the boats given in the optimal solution?

16. A trucking company with Rs 40,00,000 to spend on new equipment is contemplating three types of vehicles. Vehicle *A* has a 10-tonne pay-load and is expected to average 35 km per hour. It costs Rs 80,000. Vehicle *B* has a 20-tonne pay-load and is expected to average 30 km per hour. It costs

Find out the optimal number of advertisements which the company should release for the three magazine so as to have the maximum exposures among the prospective buyers. How many such persons can be reached through the magazine advertisement?

30. Solve the following LPP:

Minimise $Z = 120x_1 + 60x_2$

Subject to

$$20x_1 + 30x_2 \geq 900$$
$$40x_1 + 30x_2 \geq 1200$$
$$x_1, x_2, \geq 0$$

What would happen if the objective function of this problem was of the 'maximisation' type?

31. Two products *A* and *B* are processed on three machines M_1, M_2 and M_3. The processing times per unit, machine availability and profit per unit are:

Machine	Processing time (hours)		Availability (hours)
	A	B	
M_1	2	3	1,500
M_2	3	2	1,500
M_3	1	1	1,000
Profit	Rs 10	Rs 12	

Any unutilized time on machine M_3 can be given on rental basis to another firm at an hourly rate of Rs 1.50. Formulate the mathematical model, solve it using simplex technique to determine the maximum profit that can be made using the resources. What is the total profit earned?

32. A manufacturer produces three commodities, C_1, C_2 and C_3, which require, respectively, 2, 2 and 3 units of material and, respectively, 2, 2 and 1 hours of labour. The unit profit on C_1, C_2 and C_3 is Rs 6, Rs 3 and Rs 2 in that order.

In a given week, if 300 units of material and 120 hours of labour are available, what product mix should he make in order to maximise the profit?

Would the answer be different if the manufacturer decides to discontinue the production of the least profitable commodity, C_3? If yes, what would be the new product mix?

33. Given below are the objective function, the constraints, and the final simplex tableau for a linear programming product-mix problem:

Objective function:

Maximise $Z = 2x_1 + 5x_2 + 8x_3$

Constraints:

$$6x_1 + 8x_2 + 4x_3 \leq 96 \quad \text{(hours, department I)}$$
$$2x_1 + x_2 + 2x_3 \leq 40 \quad \text{(hours, department II)}$$
$$5x_1 + 3x_2 + 2x_3 \leq 60 \quad \text{(hours, department III)}$$
$$x_1, x_2, x_3 \geq 0$$

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Chapter Overview

For every linear programming problem, there is a mirror image problem, which is also a linear programming problem. This is termed the 'primal-dual' relationship. Since the two problems are connected to each other, there is obviously a relationship to be expected between their solutions. There indeed is. This chapter explores and explains this and does more. Apart from a mathematical connection between primal and dual, it explains the economic significance of the dual and helps a manager to answers to questions like the following:

- *What is the marginal profitability of each of the resources of the firm? In turn, it means by how much the profit will increase if more quantity of a particular resource is added or how much reduction in profit will result if its availability is reduced.*
- *To what extent additional quantities of a resource (or its reduction) will cause an increase (or decrease) at a uniform rate?*
- *Will the optimal product-mix need to change if the profitability of the various products changes? Obviously, when some small changes (increases or decreases) occur in the profitability of the products, it would not cause changes in the quantities of the products being produced, but then what are the limits of these price changes?*
- *Is it advisable to introduce a new product given the amounts of resources required for its production and its profitability?*
- *How would the product-mix change, if at all, its technological changes cause the resource requirements of a product to change?*

The chapter demonstrates the high potential of the Simplex method in terms of not only solving the linear programming problems but also generating a lot of information useful for managers. This chapter involves handling of inequalities, arithmetical operations on fractional values, plotting of equalities and inequalities on graphs, and their understanding, and concepts of matrices and their transpose.

4.2.2 Dual of LPP with Mixed Constraints

Sometimes a given LPP has mixed restrictions so that the inequalities given are not all in the right direction. In such a case, we should convert the inequalities in the wrong direction into those in the right direction. Similarly, if an equation is given in respect of a certain constraint, it should also be converted into inequality. To understand fully, consider the following examples.

Example 4.2 Write the dual of the following LPP:

Minimise $Z = 10x_1 + 20x_2$

Subject to

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Here, the first two inequalities are in the right direction (being ≥ type with a minimisation type of objective function) while the third one is not. Multiplying both sides by – 1, this can be written as $-2x_1 + x_2 \geq -6$. Now, we can write the primal and dual as follows:

	Primal		*Dual*
Minimise	$Z = 10x_1 + 20x_2$	Maximise	$G = 18y_1 + 8y_2 - 6y_3$
Subject to		Subject to	
	$3x_1 + 2x_2 \geq 18$		$3y_1 + y_2 - 2y_3 \leq 10$
	$x_1 + 3x_2 \geq 8$		$2y_1 + 3y_2 + y_3 \leq 20$
	$-2x_1 + x_2 \geq -6$		
	$x_1, x_2 \geq 0$		$y_1, y_2, y_3 \geq 0$

Example 4.3 Obtain the dual of the LPP given here:

Maximise $Z = 8x_1 + 10x_2 + 5x_3$

Subject

$$x_1 - x_3 \leq 4$$
$$2x_1 + 4x_2 \leq 12$$
$$x_1 + x_2 + x_3 \geq 2$$
$$3x_1 + 2x_2 - x_3 = 8$$
$$x_1, x_2, x_3 \geq 0$$

We shall first consider the constraints.

Constraints 1 and 2: Since they are both of the type ≤, we do not need to modify them.

Constraint 3: This is of type ≥. Therefore, we can convert it into ≤ type by multiplying both sides by – 1 to become $-x_1 - x_2 - x_3 \leq -2$.

If we denote marginal profitability of material as MP_M and the marginal profitability of labour as MP_L, respectively, the shadow prices of the two resources, we can write the dual as follows:

Minimise $G = 60MP_M + 96MP_L$

Subject to

$$2MP_M + 4MP_L \geq 40$$
$$3MP_M + 3MP_L \geq 35$$
$$MP_M, MP_L \geq 0$$

Now let us turn to the economic significance of the surplus variables S_1 and S_2 in the dual, whose numerical values can be obtained from Δ_j row in the optimal solution of the primal. The value of S_1 in the optimal solution represents the opportunity cost of the product *A* while the value of S_2 represents that of product *B*. For the product *A*, production of an additional unit will get the firm a profit of Rs 40 and , at the same time, the firm would use up resources worth $2 \times 10/3 + 4 \times 25/3 =$ Rs 40. Thus, the net effect of producing one unit of product would be $40 - 40 = 0$. Similarly, for product *B*, the opportunity cost equals zero.

Further, the Δ_j value equal to 18 under S_1 indicates that each rupee of additional profit in product *A* would increase the minimum rent acceptable by Rs 18 while in case of product *B*, the rent shall be up by Rs 8. Similarly, a reduction in unit profit in products *A* and *B* shall reduce the rent acceptable by Rs 18 and 8, respectively.

The Minimisation Problem A minimisation problem, Example 3.3, is reproduced here.

Minimise $Z = 40x_1 + 24x_2$ Total cost

Subject to

$20x_1 + 50x_2 \geq 4{,}800$ Phosphate requirement

$80x_1 + 50x_2 \geq 7{,}200$ Nitrogen requirement

$x_1, x_2 \geq 0$

From the optimal solution given in the Table 4.6, reproduced from Table 3.13, it may be noted that the fertiliser requirement of the farmer can be met with at a minimum cost of Rs 3,456, by buying 144 bags only of mixture *B*.

Table 4.6 *Simplex Tableau: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	24	8/5	1	0	−1/50	0	1/50	144
S_1	0	60	0	1	−1	−1	1	2,400
c_j		40	24	0	0	M	M	
Solution		0	144	2,400	0	0	0	
Δ_j		8/5	0	0	12/25	M	$M - \frac{12}{25}$	

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

(b) *For Product B and C* The value of resources required for their production is as follows:

Produt B		*Product C*	
	Rs		Rs
5 hours @ Rs 2/3	= 10/3	2 hours @ Rs 2/3	= 4/3
4 hours @ Rs 5/3	= 20/3	4 hours @ Rs 5/3	= 20/3
4 hours @ Re 0	= 0	5 hours @ Re 0	= 0
Total	= Rs 10		= Rs 8
Profit per unit	= Rs 10		= Rs 8

For each of them, the opportunity cost is equal to zero.

(c) The hours in the packaging department have a zero opportunity cost because there is excess capacity in this department.

4.4 SENSITIVITY ANALYSIS

Equally important as obtaining the optimal solution to a linear programming problem is the question as to how the solution would be affected if the parameters c_j, b_i, or a_{ij} of the problem change. This question is answered by the *sensitivity analysis,* also called the *post-optimality analysis.* Of course, we may directly substitute the changed values in a given situation and re-solve the problem to determine the effect on the optimal solution. It will be appreciated, however, that it would be a lot advantageous if we could obtain this information graphically or from the tableau containing the final solution to the problem, without being required to carry out the whole exercise again. Let us consider as to how can we get the information of this type.

We shall first illustrate a graphic introduction to the sensitivity analysis and then consider the question through simplex tableau.

4.4.1 Graphic Approach to Sensitivity Analysis

Let us reconsider Example 3.1 reproduced below:

Maximise $Z = 40x_1 + 35x_2$ Profit

Subject to

$2x_1 + 3x_2 \leq 60$ Raw Material Constraint

$4x_1 + 3x_2 \leq 96$ Labour Hours Constraint

$x_1, x_2 \geq 0$

where x_1 is the number of units of product A and x_2 is the number of units of product B.

The graph depicting the constraints and iso-profit lines is shown in Figure 4.1

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

We shall discuss the sensitivity analysis in the context of Example 3.2, which is reproduced here along with its optimal solution contained in Table 4.10. Once the concept is grasped, the reader may as well verify the values obtained graphically in respect of the Example 3.1.

Maximise	$Z = 5x_1 + 10x_2 + 8x_3$	Profit
Subject to		
	$3x_1 + 5x_2 + 2x_3 \le 60$	Fabrication hours
	$4x_1 + 4x_2 + 4x_3 \le 72$	Finishing hours
	$2x_1 + 4x_2 + 5x_3 \le 100$	Packaging hours
	$x_1, x_2, x_3 \ge 0$	

Table 4.10 *Simplex Tableau: Optimal Solution*

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2	10	1/3	1	0	1/3	−1/6	0	8
x_3	8	2/3	0	1	−1/3	5/12	0	10
S_3	0	−8/3	0	0	1/3	−17/12	1	18
c_j		5	10	8	0	0	0	
Solution		0	8	10	0	0	18	
Δ_j		−11/3	0	0	−2/3	−5/3	0	

In context of this problem, we now consider the following:

(a) Changes in Objective Function Coefficients, c_j's

We first consider the question as to how the changes in the coefficients of the decision variables in the objective function, the profit rates in our case, shall influence the optimal solution. In the context of our problem, the variable x_1 is not in the solution—it is a non-basic variable, while the variables x_2 and x_3 are in the basis. We proceed in respect of each of these two types of variables, as follows.

(i) Variables Included in the Solution Consider whether the change of profit per unit of a product, that is currently being produced, causes a change in the optimal solution to the problem. Over a certain range, a change, positive or negative, in the unit profit would not cause a change in the optimal solution. To determine this, we divide the Δ_j row values (of the optimal solution) by the corresponding row values in the tableau in respect of the variable in question. For our example, in respect of x_2, we have,

Δ_j	:	−11/3	0	0	−2/3	−5/3	0
Coefficients in row x_2	:	1/3	1	0	1/3	−1/6	0
Quotient	:	−11	0	–	−2	10	
					↑	↑	
					Least negative	Least positive	

sation of this concept. According to this rule, in case of multiple changes in the objective function coefficients, if the summation of all the r_j's does not exceed 1 (i.e. 100 percent), then the current basis shall remain unaltered, while if it exceeded 1, then the current basis may, or may not, hold valid. Of course, although the basis would not change when $\Sigma r_j \leq 1$, the z – value may change due to changes in the coefficient values.

Example 4.8 For Example 4.6, determine whether the current basis would change for each of the following cases:

(a) The profit per unit of P_2 increases to Rs 85, while profit per unit of P_3 declines to Rs 70. What is the new optimal value of Z?

(b) The profit per unit of P_1, P_2 and P_3 change as follows:

P_1 : Rs 80 P_2 : Rs 77.5 P_3 : Rs 60

(a) Here, $\Delta c_2 = 85 - 65 = 20$

With $I_2 = 41.67$, we have $r_2 = 20/41.67 = 0.48$

Also, $\Delta c_3 = 70 - 80 = -10$

With $D_3 = 31.25$, we have, $r_3 = -(-10)/31.25 = 0.32$

Now, since r_1 and $r_4 = 0$, we get

$$\Sigma r_j = 0 + 0.48 + 0.32 + 0 = 0.80 \text{ or } 80\%.$$

Since the total is less than 100%, the current basis remains unchanged.

The new z – value is: $70 \times 0 + 85 \times 15 + 70 \times 10 + 75 \times 0 = \text{Rs } 1,975.$

(b) From the given changes:

$\Delta c_1 = 80 - 70 = 10$ $I_1 = 57.50$ $r_1 = 10/57.50 = 0.17$

$\Delta c_2 = 77.5 - 65 = 12.5$ $I_2 = 41.67$ $r_2 = 12.0/41.67 = 0.30$

$\Delta c_3 = 60 - 80 = -20$ $D_3 = 31.25$ $r_3 = 20/31.25 = 0.64$

$\Delta c_4 = 0$ $r_4 = 0$

Here, $\Sigma r_j = 0.17 + 0.30 + 0.64 + 0.00 = 1.11$. Since the total exceeds unity, we cannot be sure whether the combined changes would affect optimal solution or not.

(ii) Changes in the RHS Values

The 100% Rule can be applied also when changes in the right-hand-side values are considered. In this case also, there are two cases to be investigated into. They are:

1. When *all* the constraints whose right-hand-side values are changed are non-binding constraints.
2. When at *least one* constraint whose right-hand-side values are changed is a binding constraint.

Recall that a binding constraint is one whose LHS exactly matches the RHS, so that it has a solution value equal to zero.

In the first case, the current basis remains optimal if each of the right-hand-side value remains within its allowable limits. In such a situation, the values of the decision variables and the optimal value of Z remain unchanged. However, if the right-hand-side value of any constraint is changed outside its allowable range, the current basis would no longer be optimal.

Example 4.9 Does the current basis remain optimal in each of the following cases, in context of the Example 4.6?

(a) Raw material R_2 reduced to 100 kg and labour L_2 increased to 150 hours.

(b) Supply of raw material R_2 increased to 160 kg and labour L_2 supply reduced to 80 hours.

Table 4.15 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	2	5	10	1	0	0	900	450
S_2	0	2	5	3	0	1	0	400	200
S_3	0	4*	2	2	0	0	1	600	150 ←
c_j		40	30	20	0	0	0		
Solution		0	0	0	900	400	600		Z = 0
Δ_j		40	30	20	0	0	0		
		↑							

Table 4.16 *Simplex Tableau 2: Non-optimal Solution*

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	0	4	9	1	0	–1/2	600	150
S_2	0	0	4*	2	0	1	–1/2	100	25 ←
x_1	40	1	1/2	1/2	0	0	1/4	150	300
c_j		40	30	20	0	0	0		
Solution		150	0	0	600	100	0		Z = 6,000
Δ_j		0	10	0	0	0	–10		
			↑						

Table 4.17 *Simplex Tableau 3: Optimal Solution*

Basis		x_1	x_2	x_3	S_1	S_2	S_3	b_i
S_1	0	0	0	7	1	–1	0	500
x_2	30	0	1	1/2	0	1/4	–1/8	25
x_1	40	1	0	1/4	0	–1/8	5/16	137.5
c_j		40	30	20	0	0	0	
Solution		137.5	25	0	500	0	0	Z = 6,250
Δ_j		0	0	–5	0	–5/2	–35/4	

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Example 4.14 The simplex tableau for a maximisation problem of linear programming is given as follows:

Product Mix c_j	x_j	x_1	x_2	S_1	S_2	Quantity (b_i)
5	x_2	1	1	1	0	10
0	S_2	1	0	–1	1	3
	c_j	4	5	0	0	
	z_j	5	5	5	0	50
	$c_j - z_j$	– 1	0	– 5	0	

Answer the following questions, giving reasons in brief:

(a) Is this solution optimal?

(b) Are there more than one optimal solutions?

(c) Is this solution degenerate?

(d) Is this solution feasible?

(e) If S_1 is slack in machine *A* (in hours/week) and S_2 is slack in machine *B* (in hours/week), which of these machines is being used to the full capacity when producing according to this solution?

(f) A customer would like to have one unit of product x_1 and is willing to pay in excess of the normal price in order to get it. How much should the price be increased in order to ensure no reduction of profits?

(g) How many units of the two products x_1 and x_2 are being produced according to this solution and what is the total profit?

(h) Machine *A* (associated with slack S_1, in hours/week) has to be shut down for repairs for 2 hours next week. What will be the effect on profits?

(i) How much would you be prepared to pay for another hour (per week) of capacity each on machine *A* and machine *B*?

(j) A new product is proposed to be introduced which would require processing time of 1/2 hour on machine *A* and 20 minutes on machine *B*. It would yield a profit of Rs 3 per unit. Do you think it is advisable to introduce this product?

The given simplex tableau is reproduced in Table 4.21.

Table 4.21 *Simplex Tableau*

Basis		x_1	x_2	S_1	S_2	b_i
x_2	5	1	1	1	0	10
S_2	0	1	0	– 1	1	3
c_j		4	5	0	0	
Solution		0	10	0	3	
Δ_j		– 1	0	– 5	0	

The manager of the company insists that the products P_3 and P_4 should be given top priority in the production, because they yield the maximum unit contribution.

Formulate this as a linear programming problem, obtain the optimal solution and state whether you agree with the manager's viewpoint. Further, attempt the following.

(a) Write the dual of this problem.
(b) Obtain the marginal profitability of (i) raw materials, (ii) the time in the production department, and (iii) the time in the finishing and packaging department.
(c) Over what range of values of the respective constraints the marginal profitabilities determined by you in (b) earlier would be valid?
(d) Obtain the optimal values of the dual variables.
(e) Verify that the objective function values of the primal and the dual problems are identical.
(f) The marketing manager informs that the selling price of the product P_2 has to be revised downward to Rs 116. What would be the new optimal product mix and the profit at it?

Let x_1, x_2, x_3, x_4 and x_5 be the number of units of P_1, P_2, P_3, P_4 and P_5, respectively, to be produced. The LPP corresponding to the given information can be expressed as:

Maximise $Z = 150x_1 + 120x_2 + 160x_3 + 160x_4 + 100x_5$

Subject to

$$10x_1 + 10x_2 + 20x_3 + 30x_4 + 20x_5 \le 50{,}000$$
$$10x_1 + 20x_2 + 10x_3 + 10x_4 + 20x_5 \le 80{,}000$$
$$30x_1 + 20x_2 + 20x_3 + 20x_4 + 10x_5 \le 140{,}000$$
$$x_i \ge 0,\ i = 1, 2, \ldots, 5$$

After introducing the slack variables we get,

Maximise $Z = 150x_1 + 120x_2 + 160x_3 + 160x_4 + 100x_5 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$10x_1 + 10x_2 + 20x_3 + 30x_4 + 20x_5 + S_1 = 50{,}000$$
$$10x_1 + 20x_2 + 10x_3 + 10x_4 + 20x_5 + S_2 = 80{,}000$$
$$30x_1 + 20x_2 + 20x_3 + 20x_4 + 10x_5 + S_3 = 140{,}000$$
$$x_i \ge 0,\ i = 1, 2, \ldots, 5,\ S_1, S_2, S_3 \ge 0$$

The solution is contained in Tables 4.24 through 4.27.

Table 4.24 *Simplex Tableau 1 : Non-optimal Solution*

Basis		x_1	x_2	x_3	x_4	x_5	S_1	S_2	S_3	b_i	b_i/a_{ij}
S_1	0	10	10	20*	30	20	1	0	0	50,000	2,500 ←
S_2	0	10	20	10	10	20	0	1	0	80,000	8,000
S_3	0	30	20	20	20	10	0	0	0	140,000	7,000
c_j		150	120	160	160	100	0	0	1		
Solution		0	0	0	0	0	50,000	80,000	140,000		
Δ_j		150	120	160	160	100	0	0	0		
				↑							

The shadow prices from the primal problem yield the solution to the dual problem so that $X = 1.0$ and $Y = 0.3$. Substitution of these values in the five constraints of the dual problem gives the following surplus variables:

$$S_1 = 0.3, S_2 = 0, S_3 = 0.95, S_4 = 0.5, \text{ and } S_5 = 0$$

The fact that there is no surplus in the second and the fifth constraints indicates that the non-zero variables in the primal problem are B and E. Therefore, from the primal problem,

$$1.5B + 1.0E = 558$$

$$0.8B + 0.4E = 267$$

Solving these equations simultaneously, we get $B = 219$ (thousand) and $E = 229.5$ (thousand).

Thus, the contribution under the optimal solution is

$$Z = 1.74 \times 219 + 1.12 \times 229.5 = 638.1 \text{ (thousand), or Rs } 638,100$$

Under the existing programme, the contribution is,

$$Z = 50 \times 2.00 + 40 \times 1.74 + 70 \times 2.50 + 60 \times 2.66 + 20 \times 1.12 = 526.6 \text{ (thousand)} = \text{Rs } 526,600.$$

Thus, contribution can be increased by Rs 111,500.

KEY POINTS TO REMEMBER

- For every LPP, there is another LPP, called as the primal and dual, respectively. When the primal is a maximisation problem, the dual is a minimisation problem and vice versa. For an m variable and n constraints primal, the dual has n variables and m constraints.
- To write the dual, it is first ensured that all its constraints are in the right direction (i.e. $\leq$ type when the objective function is of maximisation nature, and $\geq$ type when the objective function is of minimisation nature) and the variables are all non-negative. A constraint in a direction opposite to the one required is multiplied by -1 to reverse the direction, while a constraint with an "=" sign is replaced by a pair of inequalities in the opposite directions. An unrestricted variable is set equal to the difference of two non-negative variables.
- When both the conditions are satisfied, dual variables are introduced and the three matrices involved in the primal are transposed. For every unrestricted variable in the primal, the dual has an equation in one of the constraints, while for every constraint involving an equation, the dual has an unrestricted variable.
- If the primal problem has an optimal solution, then the dual also has an optimal solution with identical objective function value. If the primal problem has an unbounded solution, the dual has no feasible solution, and vice versa.
- The Δ_j values (ignoring minus signs, if any) corresponding to the slack/surplus variables of a problem represent optimal values of the dual variables.
- Also, the Δ_j values (ignoring the minus signs) corresponding to the slack/surplus variables indicate the marginal profitability or shadow prices of the resources/variables they are related to. Thus the shadow price of a given constraint of an LPP is the amount by which the optimal value of the objective function is improved if the right hand side of the constraint is increased by one unit.
- Sensitivity analysis answers questions like how would the solution of an LPP change when the coefficients in the objective function, the right-hand-side values of the constraints or the coefficients in the left hand side of the constraints change.

3. Write the dual corresponding to the following linear programming problem:

Maximise $Z = 5x + 7y$

Subject to

$$x + y \leq 4$$
$$3x + 8y \leq 26$$
$$10x + 7y \leq 35$$
and $x, y \geq 0$

Next, find dual of the dual problem and show that it is the same as the given problem.

4. Give the dual in complete mathematical form for the following primal of a linear programming problem:

$$x_1 \leq 4;\ x_2 \leq 6;\ x_1 + x_2 \leq 5;\ -x_2 \leq -1 \text{ and } x_1, x_2 \geq 0$$

Maximise $Z = 3x_1 - 2x_2$ *(ICWA, June, 1985)*

5. Using dual, convert the following problem into a maximisation problem:

Minimise $Z = 2x_1 + 9x_2 + 3x_3$

Subject to

$$x_1 + 4x_2 + 2x_3 \geq 5$$
$$3x_1 + x_2 + 2x_3 \geq 4$$
$x_1, x_2 \geq 0$, x_3 unrestricted in sign

6. Write the dual of the following linear programming problem:

Maximise $Z = 3x_1 + 4x_2 + 7x_3$

Subject to

$$x_1 + x_2 + x_3 \leq 10$$
$$4x_1 - x_2 - x_3 \geq 15$$
$$x_1 + x_2 + x_3 = 7$$
$x_1, x_2 \geq 0$, x_3 unrestricted *(M Com., Delhi, 1982)*

7. Given:

Minimise $Z = 4x_1 + x_2$

Subject to

$$3x_1 + x_2 = 2$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
and $x_1, x_2 \geq 0$

(a) Write the dual problem for this linear programme.

(b) Solve it for the optimal values of x_1 and x_2.

8. Given the following problem:

Maximise $Z = 2x_1 + 3x_2$

Subject to

$$2x_1 + 3x_2 \leq 30$$
$$x_1 + 2x_2 \leq 10$$
$$x_1 - x_2 \leq 0$$
$$x_1, x_2 \geq 0$$

(a) Construct the dual with y_1, y_2, y_3 as dual variables.

The fertilisers will be sold in bulk and managers have proposed the following prices per tonne:

X_1 : Rs 83; X_2 : Rs 81; and X_3 : Rs 81

The manufacturing costs of each type of fertiliser, excluding materials, are Rs 11 per tonne.

You are required to:

(a) formulate the above data into a linear programming model so that the company may maximise contribution;

(b) construct the initial tableau and state what is meant by 'slack variables' (define X_4, X_5, X_6 as the slack variables for X_1, X_2, and X_3, respectively);

(c) indicate, with explanations, which will be the 'entering variable' and 'leaving variable' in the first iteration; (you are not required to solve the model.)

(d) interpret the final matrix of the simplex solution given below:

Basic variable	X_1	X_2	X_3	X_4	X_5	X_6	Solution
X_1	1	0	3	20	−10	0	4,000
X_2	0	1	−l	−10	10	0	8,000
X_6	0	0	−0.4	−3	1	1	600
Z	0	0	22	170	40	0	284,000

(e) use the final matrix above to investigate:

(i) the effect of an increase in nitrate of 100 tonnes per month;

(ii) the effect of a minimum contract from an influential customer for 200 tonnes of X_3 per month to be supplied. *(ICMA, May 1989, Adapted)*

16. Fill in the blanks:

Primal problem			Dual problem		
Variable	Solution	Δ_j	Variable	Solution	Δ_j
x_1	8/3	0	y_1	–	–
x_2	20/3	0	y_2	–	–
x_3^*	80/3	0	y_3	–	–
x_4^*	0	4/15	y_4	–	–
x_5^*	0	1/15	y_5^*	–	–
x_6^*	160/3	0	y_6^*	–	–

*surplus variable — * slack variable

Objective Function Value = 76/3. — Objective Function Value = –.

17. The Alloy Metal Company plans to purchase at least 200 quintals of scrap metal. The company decides that the scrap metal to be purchased must contain at least 100 quintals of a valuable metal M_1 and no more than 35 quintals of a base metal M_2. The company can purchase the scrap metal from two suppliers in unlimited quantities with the following percentages, by weight, of M_1 and M_2.

Metal	Supplier X	Supplier Y
M_1	25%	75%
M_2	10%	20%

Subject to

$$x_1 - x_2 \leq 3/5$$
$$x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

(M Com, Kolkata, 1993)

26. You are told that the objective row of the final tableau of a linear programming solution to a contribution maximisation problem showed shadow price values of +2 and 0, respectively, in the columns for products *A* and *B*. A third product which was included in the initial formulation is shown by separate analysis to have a dual value of –3 (i.e. a deficit value if produced). Product selling prices were Rs 20, Rs 30, Rs 15 per unit respectively.
You are required to provide:
(a) An interpretation of the values +2, 0, –3 which were based on the estimated demands for the products at the given prices;
(b) As assessment of the use the information in (a) may have in product planning.
(ICMA, May, 1983, Adapted)

27. Given below is a table obtained after a few iterations using simplex method to solve a linear programming problem to maximise total contribution margin from products *A* and *B*:

		8.5	10.5	0	0	0	c_j
Mix	*Total*	x_1	x_2	S_1	S_2	S_3	
x_2	300	0	1	3/5	–2/5	0	
x_1	300	1	0	–2/5	3/5	0	
S_3	400	0	0	–1/5	–1/5	1	

Give short answers to the following questions giving reasons as well:
(i) Is the above solution optimal?
(ii) Is the above solution feasible?
(iii) Does the problem have alternative optimal solution? If so, find another optimal solution.
(iv) Write the objective function of the problem.
(v) What are the shadow prices for the three resources?
(vi) If *S* represents the slack for the production capacity constraint, how much should the company be willing to pay for each additional unit of production capacity?
(vii) S_3 represents the slack for demand constraint. If the company is able to increase its total demand by 20 units, what will be the optimal mix and total contribution margin?

28. A manufacturer produces four products, *A*, *B*, *C* and *D*, each of which is processed on three machines, *X*, *Y* and *Z*. The time required to manufacture one unit of each of the four products and capacity of each of the three are indicated in the following table:

Product	*Processing time (in hrs)*		
	Machine X	*Machine Y*	*Machine Z*
A	1.5	4	2
B	2	1	3
C	4	2	1
D	3	1	2
Capacity (hours)	550	700	200

Chapter Overview

Distribution of goods produced by a factory from various warehouses to different markets where they are required is a common problem. The transportation method is developed to deal with the transportation of goods from different sources to different destinations, given the relevant data like availability at the sources, demand at each of the destinations and the cost of shipping along each route. The method allows the manager to seek answers to the questions like the following:

- *What is the optimal way of shipping goods from various sources (warehouses) to different markets so as to minimise the total cost involved in the shipping?*
- *How to handle a situation when some routes are not available or when some units have to be necessarily transported from a particular source to a particular market?*
- *How would the shipping schedule change if some routes become cheaper /costlier?*
- *If it were possible to increase supply, which of the sources should be preferred?*
- *Instead of allowing shipping of goods only from listed sources to different destinations, if it were possible to ship goods from a destination and then from there to a further destination, how much cost can be saved? (This is what is called a transhipment problem.)*
- *If an item can be produced at different locations at varying costs and sold in different markets at different prices, then what shipping plan will yield maximum profit?*

The use of transportation method is not limited to solution of the transportation problems alone. Problems of scheduling production, controlling inventory and management of funds over different time periods illustrate some other areas which lend themselves to handling by the transportation method.

This chapter basically requires elementary arithmetic calculations. However, familiarity with summation notation, a basic knowledge of inequalities, matrices and their transpose is also necessary. While working in this chapter, master the skill of drawing of a closed path.

5.3 SOLUTION TO THE TRANSPORTATION PROBLEM

A transportation problem can be solved by two methods, using (a) simplex method, and (b) transportation method. We shall illustrate these with the help of an example.

Example 5.1 A firm owns facilities at six places. It has manufacturing plants at places *A, B* and *C* with daily production of 50, 40, and 60 units respectively. At point *D, E,* and *F* it has three warehouses with daily demands of 20 , 95, and 35 units respectively. Per unit shipping costs are given in the following table. If the firm wants to minimise its total transportation cost, how should it route its products?

		Warehouse		
		D	*E*	*F*
	A	6	4	1
Plant	*B*	3	8	7
	C	4	4	2

5.3.1 The Simplex Method

The given problem can be expressed as an LPP as follows:

Let x_{ij} represent the number of units shipped from plant *i* to warehouse *j*. With *Z* representing the total cost we can state the problem as follows.

Minimise $$Z = 6x_{11} + 4x_{12} + 1x_{13} + 3x_{21} + 8x_{22} + 7x_{23} + 4x_{31} + 4x_{32} + 2x_{33}$$

Subject to

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= 50 \\ x_{21} + x_{22} + x_{23} &= 40 \\ x_{31} + x_{32} + x_{33} &= 60 \end{aligned}\right\} \quad \text{Supply constraints}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 20 \\ x_{12} + x_{22} + x_{32} &= 95 \\ x_{13} + x_{23} + x_{33} &= 35 \end{aligned}\right\} \quad \text{Demand constraints}$$

$$x_{ij} \geq 0 \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

Now, this problem can be solved as any other problem using the *simplex algorithm.* But the solution is going to be very lengthy and a cumbersome process because of the involvement of a large number of decision and artificial variables. Hence, we look for an alternate solution procedure called the *transportation method* which is an efficient one that yields results faster and with less computational effort. A significant point of difference between the simplex and the transportation methods is regarding the determination of initial basic feasible solution. As we use simplex method to solve a minimisation problem, we must add artificial variables to make the solution artificially feasible. As we progress from one tableau to another, the artificial variables are dropped one after the other as they become non-basic. Then, eventually an optimal solution is obtained where all the artificial variables are excluded. The transportation method obviates the need to use artificial variables because with this method it is fairly easy to find an initial solution that is feasible, without using the artificial variables.

cell chosen has minimum cost. To illustrate, suppose the largest cost difference is found to be tied for a row and a column. In applying first of the rules, determine the quantity (considering supply availability and demand) which can be assigned in each case and select the one where larger quantity can be assigned. In the other case, compare unit cost values of the two least cost cells, in the row and the column, and choose the one which has a lower value.

Delete the column/row which has been satisfied. Again, find out the differences and proceed in the same manner as discussed earlier. Continue until all units have been assigned. The VAM is shown schematically in Fig. 5.2.

For Example 5.1, the initial feasible solution by using this method is given in Table 5.5.

The differences between the two least-cost cells are calculated for each row and column. The largest of these being 4(= 7 – 3), the row designated as *B* is selected. In the lowest cost cell of the row, *BD*, a value 20 is assigned and the column *D* is deleted as the demand is satisfied. In the second iteration, again the cost differences are calculated and the first row is selected as it shows the greatest difference value of 3. In the cell *AF*, 35 is assigned and column *F* is deleted. Now, only one column is left and therefore no differences need be calculated. The assignments can be easily made having reference to the supply at various origins. The assignment made by VAM involves a total cost of Rs 555—a solution same as that by LCM, and better than the one obtained by NW corner rule involving a total cost of Rs 730.

Table 5.5 *Initial Basic Feasible Solution: VAM*

To → / From ↓	D	E	F	Supply	Iteration I	Iteration II
A	6	4 [15]	1 [35]	~~50~~ 15	3	(3)
B	3 [20]	8 [20]	7	~~40~~ 20	(4)	1
C	4	4 [60]	2	60	2	2
Demand	20	95	35	150		
I	1	0	1			
II	–	0	1			

Total cost: $4 \times 15 + 1 \times 35 + 3 \times 20 + 8 \times 20 + 4 \times 60 =$ Rs 555

The Vogel's approximation method is also called the *Penalty Method* because the cost differences that it uses are nothing but the penalties of not using the least-cost routes. Since the objective function is the minimisation of the transportation cost, in each iteration that route is selected which involves the maximum penalty of *not* being used.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

This plan also has 5 (= 3 + 3 – 1) allocations. It involves a total cost of Rs 590. We will again apply step 2 to determine whether this solution is optimal or not, and then apply step 3 if it is found to be non-optimal. Application of step 2 yields the following results.

Cell	*Closed Loop*	*Net Cost Change*	*Opportunity Cost*
AD	*AD-AE-BE-BD*	+6 – 4 + 8 – 3 = +7	–7
AF	*AF-CF-CE-AE*	+1 – 2 + 4 – 4 = –1	+1
BF	*BF-CF-CE-BE*	+7 – 2 + 4 – 8 = +1	–1
CD	*CD-BD-BE-CE*	+4 – 3 + 8 – 4 = +5	–5

Since only the cell *AF* has a positive opportunity cost, we shall include it in the transportation schedule. The maximum number of units that can be routed through *AF* is 35. Accordingly, the revised solution is given in Table 5.8.

Table 5.8 *Improved Solution: Optimal*

From ↓ / *To* →	*D*	*E*	*F*	*Supply*
A	6	4 [15]	1 [35]	50
B	3 [20]	8 [20]	7	40
C	4	4 [60]	2	60
Demand	20	95	35	150

Total cost: 4 × 15 + 1 × 35 + 3 × 20 + 8 × 20 + 4 × 60 = Rs 555

This solution also involves 5 assignments at a total cost of Rs 555. This solution would now be tested for optimality. This is done here:

Cell	*Closed Loop*	*Net Cost Change*	*Opportunity Cost*
AD	*AD-AE-BE-BD*	+6 – 4 + 8 – 3 = +7	–7
BF	*BF-BE-AE-AF*	+7 – 8 + 4 – 1 = +2	–2
CD	*CD-BD-BE-CE*	+4 – 3 + 8 – 4 = +5	–5
CF	*CF-CE-AE-AF*	+2 – 4 + 4 – 1 = +1	–1

Since the opportunity costs of all the empty cells are negative, the solution obtained is the optimal one.

In a similar fashion, we can test the initial basic feasible solution obtained by LCM or VAM. If we compare it with the solution contained in Table 5.8 we find that the two are identical. Clearly, therefore, the solution we obtained by VAM is optimal.

Before we discuss the MODI method for testing the optimality of a transportation solution, a few words on the tracing of a closed loop follow.

Here the cell (2, 1) has the largest positive opportunity cost and therefore we select x_{21} for inclusion as a basic variable. The closed loop starting with this cell has been shown in the table. The revised solution is shown in Table 5.10.

Table 5.10 *Improved Solution: Non-optimal*

From ↓ / To →	*D*	*E*	*F*	*Supply*	u_i
A	6 (−7)	− 4 [50]	+ 1 (1)	50	0
B	3 [20]	8 [20]	7 (−1)	40	4
C	4 (−5)	+ 4 [25]	− 2 [35]	60	0
Demand	20	95	35	150	
v_j	−1	4	2		

This solution is tested for optimality and is found to be non-optimal. Here the cell (1, 3) has a positive opportunity cost and therefore a closed loop is traced starting with this. The solution resulting is shown in Table 5.11. This, when tested, is found to be optimal, involving a total transportation cost of Rs 555.

Table 5.11 *Improved Solution: Optimal*

From ↓ / To →	*D*	*E*	*F*	*Supply*	u_i
A	6 (−7)	4 [50]	1 [35]	50	0
B	3 [20]	8 [20]	7 (−2)	40	4
C	4 (−5)	4 [60]	2 (−1)	60	0
Demand	20	95	35	150	
v_j	−1	4	1		

Table 5.15 *Alternate Optimal Solution*

Warehouse	Market A	Market B	Market C	Supply	u_i
1	M	12 (−1)	7 [180]	180	15
2	14 [60]	11 (−1)	6 [40]	100	14
3	9 [80]	5 [80]	M	160	9
4	11 (0)	7 [120]	9 (−6)	120	11
5	0 [100]	0 (−4)	0 (−8)	100	0
Demand	240	200	220	660	
v_j	0	−4	−8		

5.4.4 Degeneracy

We have already seen that a basic feasible solution of a transportation problem has $m + n - 1$ basic variables, which means that the number of occupied cells in such a solution is *one* less than the number of rows plus the number of columns. It may happen sometimes that the number of occupied cells is less than $m + n - 1$. Such a solution is called a *degenerate solution.*

Degeneracy in a transportation problem can figure in two ways. The problem may become degenerate in the first instance when an initial feasible solution is obtained (see Example 5.3). Secondly, the problem may become *degenerate* when two or more cells are vacated simultaneously in the process of transferring units along the closed path.

The problem, when a solution is degenerate, is that it cannot be tested for optimality. Both, the stepping stone method and the modified distribution method (MODI) are inoperative in such a case. The former cannot be applied because for some of the empty cells the closed loops cannot be traced, while the latter fails because all u_i and v_j values cannot be determined.

To overcome the problem, we proceed by assigning an infinitesimally small amount, close to zero, to one (or more if the need be) empty cell and treat the cell as an occupied cell. This amount is represented by a Greek letter ε (epsilon) and is taken to be such an insignificant value that would not affect the total cost. Thus, it is big enough to cause the particular route to which it is assigned to be considered as a basic variable but not large enough to cause a change in the total cost and the other non-zero amounts. Although ε is, theoretically, non-zero, the operations with it in the context of problem at hand are given here:

$$k + \varepsilon = k; \qquad \varepsilon + \varepsilon = \varepsilon;$$
$$k - \varepsilon = k; \qquad \varepsilon - \varepsilon = 0; \quad \text{and}$$
$$0 + \varepsilon = \varepsilon; \qquad k \times \varepsilon = 0.$$

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Iteration 2

Table 5.20 *Revised Solution: Non-optimal*

To → / From ↓	1	2	3	4	Supply	u_i
A	7 [20] (−)	3 [ε] (+)	8 (−2)	6 [40]	60	0
B	4 (2) (+)	2 [50] (−)	5 [50]	10 (−5)	100	−1
C	2 (0)	6 (−8)	5 (−4)	1 [40]	40	−5
Demand	20	50	50	80	200	
v_j	7	3	6	6		

Total cost: 7 × 20 + 6 × 40 + 2 × 50 + 5 × 50 + 1 × 40 = Rs 770

Iteration 3

Table 5.21 *Revised Solution: Optimal*

To → / From ↓	1	2	3	4	Supply	u_i
A	7 (−2)	3 [20]	8 (−2)	6 [40]	60	0
B	4 [20]	2 [30]	5 [50]	10 (−5)	100	−1
C	2 (−2)	6 (−8)	5 (−4)	1 [40]	40	−5
Demand	20	50	50	80	200	
v_j	5	3	6	6		

Total cost: 3 × 20 + 6 × 40 + 4 × 20 + 2 × 30 + 5 × 50 + 1 × 40 = Rs 730

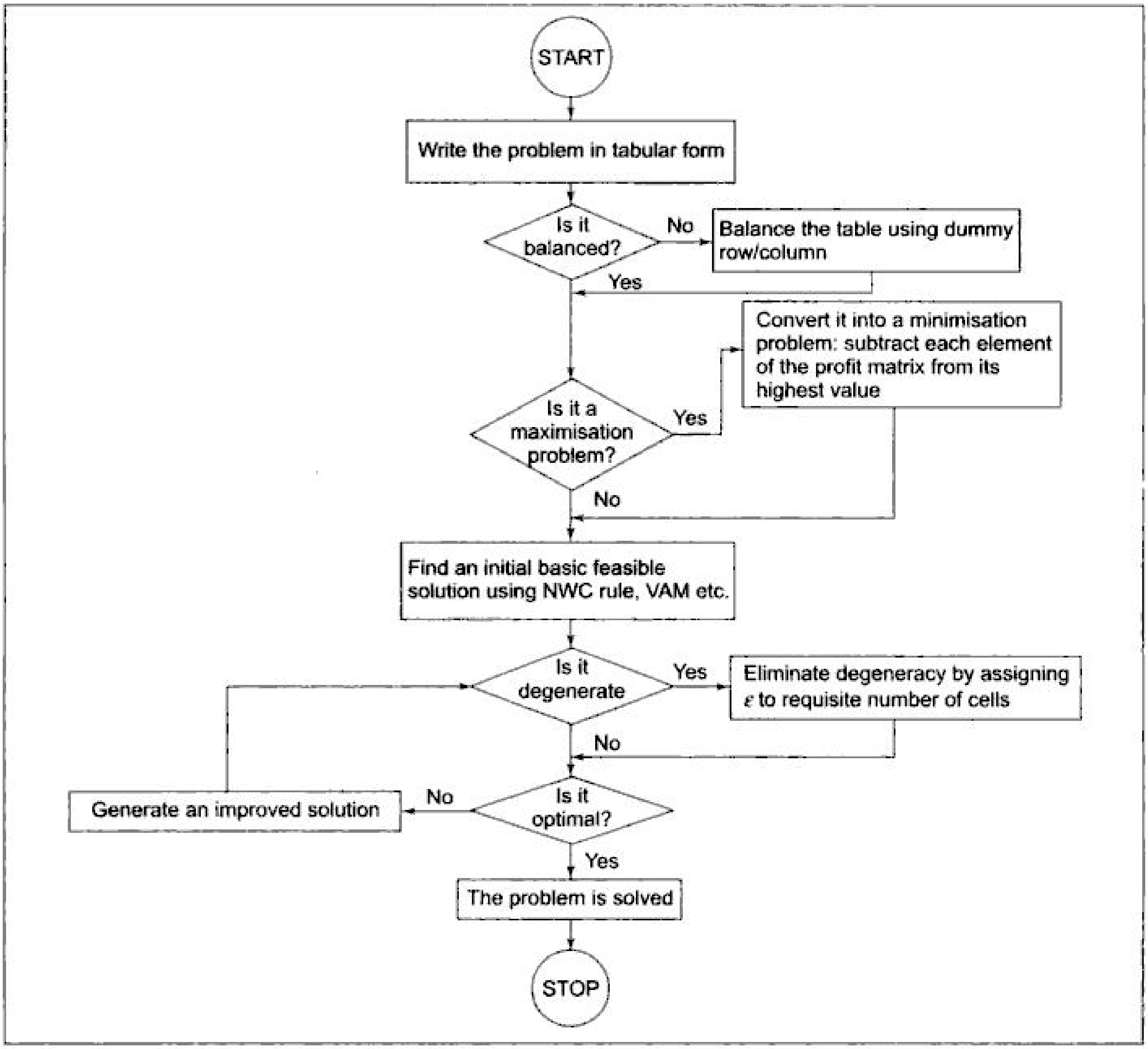


Fig. 5.4 Schematic Presentation of Transportation Method

5.5 DUAL OF THE TRANSPORTATION MODEL

As already observed, the transportation problem is only a special case of linear programming problems. It has, therefore, a dual with the usual pricing interpretations, as discussed in the previous chapter. As we have seen, a transportation problem can be written as a LPP as follows:

Minimise $$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i$$

Now, the problem can be solved as a transportation problem.

The solution is left as an exercise for the reader. Table 5.27 contains final solution to the problem, that is an optimal one. Accordingly, the production schedule, involving a total cost of Rs 1885, is as given in Table 5.28.

Table 5.27 *Final, Improved Solution: Optimal*

Origin	Destination					Supply
	1	2	3	4	D	
R_1	6 [50]	9 [10]	12	15	0	60
O_1	10	13	16	19	0 [20]	20
R_2	M	6 [60]	9	12	0	60
O_2	M	10	13 [5]	16	0 [15]	20
R_3	M	M	6 [60]	9	0	60
O_3	M	M	10 [20]	13	0	20
R_4	M	M	M	6 [60]	0	60
O_4	M	M	M	10 [15]	0 [5]	20
Demand	50	70	85	75	40	320

Table 5.28 *Production Schedule*

Produce in Month		*Units*	*For use in Month*			
			1	*2*	*3*	*4*
1	R	60	50	10		
	O.T.	nil				
2	R	60		60		
	O.T.	5			5	
3	R	60			60	
	O.T.	20			20	
4	R	60				60
	O.T.	15				15
Demand			50	70	85	75

Table 5.32 *Improved Solution: Non-optimal*

Terminal	*A*	*B*	*C*	*D*	*E*	*F*	*Supply*	u_i
A	0 [−0]	(−6) 3	(−4) 2	(−6) 6	(−2) 4	1 [50]	50	0
B	(0) 3	0 [−0]	(−5) 4	3 [40]	(−3) 8	(−3) 7	40	3
C	(0) 2	(−5) 4	0 [−0]	(−2) 4	− 4 [60]	(1) + 2	60	2
D	(−6) 6	(−6) 3	(−6) 4	0 [−20]	2 [20]	(−4) 5	0	0
E	(−6) 4	(−13) 8	(−8) 4	(−4) 2	0 [−0]	(−2) 1	0	−2
F	(−2) 1	(−11) 7	(−5) 2	(−6) 5	+ 1 [15]	− 0 [−15]	0	−1
Demand	0	0	0	20	95	35	150	
v_j	0	−3	−2	0	2	1		

Total cost: 1 × 50 + 3 × 40 + 4 × 60 + 2 × 20 + 1 × 75 = Rs 465

Table 5.33 *Improved Solution: Optimal*

Terminal	*A*	*B*	*C*	*D*	*E*	*F*	*Supply*	u_i
A	0 [−0]	(−6) 3	(−3) 2	(−6) 6	(−2) 4	1 [50]	50	0
B	(0) 3	0 [−0]	(−2) 4	3 [40]	(−3) 8	(−3) 7	40	3
C	(−1) 2	(−6) 4	0 [−0]	(−3) 4	(−1) 4	2 [60]	60	1
D	(−6) 6	(−6) 3	(−5) 4	0 [−20]	2 [20]	(−4) 5	0	0
E	(−6) 4	(−13) 8	(−7) 4	(−4) 2	0 [−0]	(−2) 1	0	−2
F	(−2) 1	(−11) 7	(−4) 2	(−6) 5	1 [75]	0 [−75]	0	−1
Demand	0	0	0	20	95	35	150	
v_j	0	−3	−1	0	2	1		

Total cost = 1 × 50 + 3 × 40 + 2 × 60 + 2 × 20 + 1 × 75 = Rs 405

Table 5.37 *Improved Solution: Optimal*

To → / From ↓	1	2	3	4	5	Supply	u_i
1	(−36) 80	(−7) 69	103 [2]	64 [2]	61 [8]	12	0
2	(−34) 47	(−69) 100	72 [16]	(−32) 65	(−10) 40	16	−31
3	16 [16]	(−69) 103	(−12) 87	36 [4]	(−61) 94	20	−28
4	(−89) 86	15 [8]	(−1) 57	(−2) 19	(−11) 25	8	−47
5	(−25) 27	20 [6]	(−11) 72	(−72) 94	19 [2]	8	−42
Demand	16	14	18	6	10	64	
v_j	44	62	103	64	61		

Total cost: $103 \times 2 + 64 \times 2 + 61 \times 8 + 72 \times 16 + 16 \times 16 + 36 \times 4 + 15 \times 8 + 20 \times 6 + 19 \times 2 =$ Rs 2,652

Example 5.8 A company has three plants and four warehouses. The supply and demand in units and the corresponding transportation costs are given.

The table (5.38) has been taken from the solution procedure of the transportation problem.

Table 5.38 *Transportation Problem: A Solution*

Plants	*Warehouses*				*Supply*
	I	*II*	*III*	*IV*	
A	5	10	4 [10]	5	10
B	6 [20]	8	7	2 [5]	25
C	4 [5]	2 [10]	5 [5]	7	20
Demand	25	10	15	5	55

Table 5.41 *Initial Feasible Solution: Optimal*

From ↓ / To →	E	F	G	R	Availability	u_i
A	(0) 3	3 [200]	(−1) 3	(−2) 6	200	0
B	4 [100]	(−1) 5	3 [200]	5 [100]	400	1
C	4 [250]	4 [50]	(−1) 4	(−1) 6	300	1
D	(−3) 5	2 [200]	(−2) 3	(−4) 7	200	−1
Requirement	350	450	200	100	1100	
v_j	3	3	2	4		

Total cost: $3 \times 200 + 4 \times 100 + 3 \times 100 + 5 \times 100 + 4 \times 250 + 4 \times 50 + 2 \times 200 = 3{,}700$ (thousand) rupees

According to the solution obtained,

For Unit	*Transfer*
A	200 to *F*
B	100 to *E*, 200 to *G*, and 100 lay off
C	250 to *E*, 50 to *F*
D	200 to *F*

(b) The solution obtained is indeed optimal. However, it is not unique for the reason that a 0 (zero) appears in one of the Δ_{ij} values calculated, in the cell *AE*. This implies that another solution exists which is as efficient as this one. The alternative solution can be obtained by changing numbers along the path ***AE-AF-CF-CE***. This would be:

For Unit	*Transfer*
A	200 to *E*
B	100 to *E*, 200 to *G*, and 100 lay off
C	50 to *E*, 250 to *F*
D	200 to *F*

(c) Since the initial solution is optimal, the cost of each is Rs 37,00,000.

(i) The initial solution to the given problem, using VAM, is given in Table 5.46.

Table 5.46 *Initial Solution: VAM*

Factory	Distribution Centers W	X	Y	Z	Supply	u_i
A	10 (−14)	8 (−9)	5 [7,000]	4 (−6)	7,000	0
B	7 (−1)	9 [6,000]	15 [1,000] −	8 [1,000] +	8,000	10
C	6 [6,000]	10 (−1)	14 (1) +	8 [4,000] −	10,000	10
Demand	6,000	6,000	8,000	5,000	25,000	
v_j	−4	−1	5	−2		

This solution is tested for optimality and found to be non-optimal. An improved solution is given in Table 5.47. This solution is found to be optimal.

The optimal solution is indicated below:

From ↓ To →	*W*	*X*	*Y*	*Z*
A			7,000	
B		6,000		2,000
C	6,000		1,000	3,000

Total cost = $5 \times 7{,}000 + 9 \times 6{,}000 + 8 \times 2{,}000 + 6 \times 6{,}000 + 14 \times 1{,}000 + 8 \times 3{,}000$ = Rs 179,000

Table 5.47 *Improved Solution: Optimal*

Factory	Distribution Centers W	X	Y	Z	Supply	u_i
A	10 (−13)	8 (−8)	5 [7,000]	4 (−5)	7,000	0
B	7 (−1)	9 [6,000]	15 (−1)	8 [2,000]	8,000	9
C	6 [6,000]	10 (−1)	14 [1,000]	8 [3,000]	10,000	9
Demand	6,000	6,000	8,000	5,000	25,000	
v_j	−3	0	5	−1		

Table 5.52 *Opportunity Loss Matrix: Optimal Solution*

Factory	*Sales Agency* 1	2	3	4	*Supply*	u_i
A	6 (−3)	2 [10]	0 (−2)	5 (−1)	10	0
B	7 (0)	7 (−1)	3 (−1)	8 [15]	15	4
C	9 [8]	9 (−1)	4 [12]	12 (−2)	20	6
D	12 (−2)	9 [2]	5 [6]	11 [7]	15	7
Demand	8	12	18	22	60	
v_j	3	2	−2	4		

We may now express the optimal solution as follows:

From: Factory	*To: Sales Agency*	*Units*	*Profit (Loss)*
A	2	10	70
B	4	15	15
C	1	8	0
	3	12	60
D	2	2	0
	3	6	24
	4	7	(14)
		Total	155

It may be noted here that the optimal solution given above is not unique. Since Δ_{ij} 0 for the cell *B*1, the problem has multiple optimal solutions.

Example 5.14 The *XYZ* Tobacco Company purchases tobacco and stores in warehouses located in the following four cities:

9. The initial solution obtained by the least-cost-method would invariably be optimal.
10. The transportation method essentially uses the same steps as of the Simplex method.
11. To determine u_i and v_j values, an initial value has to be supplied. Different initial values would lead to different u_i and v_j values, and consequently, to different incoming variables.
12. In a non-optimal solution to a transportation problem, a certain cell with $i = 2$ and $j = 3$, has $\Delta_{23} = 4$. This implies that sending a unit from source 2 to destination 3 shall save a cost of 4.
13. A closed loop would always involve an even number of cells, subject to a minimum of 4.
14. The maximum number of units which can be transferred along the closed path is equal to the *minimum* quantity chosen from among the cells bearing a negative sign on the closed path.
15. The u_i and v_j values may be determined by initially inserting any finite number which may be positive, negative or zero, to a row/column.
16. Units sent from a dummy source to various markets represent the shortfall in supply to those markets.
17. The summation of the products of the u_i values with the corresponding supply column values and of the v_j values with the corresponding demand row values would be equal to the total cost of transportation.
18. A transportation problem solution is said to be degenerate if the number of occupied cells is smaller than the number of rows *plus* the number of columns *minus* 1 (one).
19. Once non-optimal degenerate solution is obtained, the next solution is bound, also, to be degenerate.
20. In improving a non-optimal solution to a transportation problem, it is possible that more than one cell may get vacated, although normally, only one cell gets vacated and gets filled.
21. A degenerate solution may or may not be optimal.
22. To remove degeneracy, an infinitesimally small quantity is placed in each of the required number of independent cells.
23. Multiple optimal solutions are indicated if there are multiple zeros for u_i and v_j values.
24. If each cost element in a transportation problem is increased by a constant amount, it will not affect the optimal solution to the problem.
25. For an optimal solution to a transportation problem, the u_i and v_j values represent the optimal values of the dual problem.
26. If all the cost elements, c_{ij}, are multiplied by a constant, the total cost of transportation in the optimal solution shall be multiplied by the same constant.
27. A cost reduction by an amount greater than the absolute value of Δ_{ij} for a given cell would make that route a preferable one.
28. If a constant is subtracted from each value of the matrix of a profit-maximising transportation problem, it is converted into a "minimisation problem."
29. A transhipment problem allows for the shipment of goods from one source to another, and from one destination to another.
30. An m-source, n-destination transportation problem, when written as a transhipment problem would have $m + n$ sources and n destinations.

EXERCISES

1. Describe the transporation problem and give its mathematical model.
2. Explain, by taking an illustration, the North-West Corner Rule, the Least Cost Method and the Vogel's Approximation Method to obtain the initial feasible solution to a transportation problem.

8. A company is spending Rs 1,200 on transportation of its units from three plants to four distribution centres. The supply and demand of units with unit cost of transportation are given as under:

Plants	Distribution centres				Supply
	1	2	3	4	
P_1	20	30	50	17	7
P_2	70	35	40	60	10
P_3	40	12	60	25	18
Demand	5	8	7	15	

What can be the maximum saving by optimal scheduling?

9. A company has three cement plants from which cement has to be transported to four distribution centres. With identical costs of production at the three plants, the only variable costs involved are transportation costs. The monthly demand at the four distribution centres and the distance from the plants to the distribution centres (in km) are given below:

Plants	Distribution centres				Monthly production (tonnes)
	W	X	Y	Z	
A	500	1,000	150	800	10,000
B	200	700	500	100	12,000
C	600	400	100	900	8,000
Monthly demand (tonnes)	9,000	9,000	10,000	4,000	

The transportation charges are Rs 10 per tonne per km. Suggest optimal transportation schedule and indicate the total minimum transportation cost.

If, for certain reasons, route from plant *C* to distribution centre *X* is closed down, will the transportation schedule change? If so, suggest the new schedule and effect on total cost. *(M Com, Delhi, 1998)*

10. A company has four factories manufacturing the same commodity, which are required to be transported to meet the demands in four warehouses. The supplies and demands as also the cost per transportation from factory to warehouse in rupees per unit of product are given in the following table:

Factory	Warehouses				Supply (units)
	X	Y	Z	W	
A	25	55	40	60	60
B	35	30	50	40	140
C	36	45	26	66	150
D	35	30	41	50	50
Demand (units)	90	100	120	140	

Warehouse	Store				
	S_1	S_2	S_3	S_4	S_5
W_1	9	12	10	10	6
W_2	5	18	12	11	2
W_3	10	M	7	3	20
W_4	5	6	2	M	8

M in this table indicates that the route is not available.

How should the company arrange to transport the units so that the transportation cost is minimised?

18. A firm is faced with a short-term cash flow problem, which would necessitate obtaining a loan from its bank. The bank loan will be used to balance the cash inflows from accounts receivable and cash outflows from accounts payable, which are estimated as follows:

	Accounts receivable (in Rs lakh)	Accounts payable (in Rs lakh)
October	18	20
November	27	32
December	35	43

It may be assumed that both accounts receivable and accounts payable are due at the end of the relevant month and that all accounts payable are to be settled by the end of the year (December). While in any month funds from accounts receivable would flow at a time such that sufficient time would be available to finance firm's own payments of the months, in October and November, the payments to be made to the suppliers could be delayed by at the most one month. However, the delay in the payments would make the firm lose the 2 per cent discount receivable for making payment within one month.

Information about the bank loan is as follows:

(i) Any amount of loan may be taken by the firm but a loan be agreed at the start of a given month.

(ii) The loan would carry an interest rate at the rate of 2.5 per cent per month and attract a minimum of one month's interest.

Further, any surplus cash may be invested with the bank, which would earn an interest 1 per cent per month.

Using the above information, formulate the problem as a transportation problem by appropriately defining the costs and flows. Also determine the optimal solution to the problem.

19. A transportation problem involving three sources and four distribution centres is presented in the following table.

You are required to answer the following questions:

(a) Is the solution feasible?

(b) Is the solution degenerate?

(c) Is the solution optimal? Is it unique? Why?

(d) What is the opportunity cost of transporting one unit from source S_1 to D_4?

(e) What are the optimal values of the dual variables to the given problem? Show that the primal and dual both have same objective function values.

(f) What do u_i and v_j values indicate?

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

The following additional information is also provided:
- *P*, *Q*, *R* and *S* represent the selected investments.
- The company has decided to have four years' investment plan.
- The policy of the company is that amount invested in any year will remain so until the end of the fourth year.
- The values (paise) in the table represent net return on investment of one Rupee till the end of the planning horizon (for example, a Rupee invested in Investment *P* at the beginning of year 1 will grow to Rs 1 .95 by the end of the fourth year, yielding a return of 95 paise).

Using the above, determine the optimum investment strategy. *(CA, November, 1996)*

25. The personnel manager of a manufacturing company is in the process of filling 175 jobs in six different entry level skills due to the establishment of a third shift by the company. Union wage scales and requirements for the skills are shown in the following table:

Pay scales and skill requirements							
Entry level skill	:	*A*	*B*	*C*	*D*	*E*	*F*
Wage scale (Rs PM)	:	1,000	1,100	1,200	1,300	1,400	1,500
No. Required	:	25	29	31	40	33	17

Two hundred and thirty applicants for jobs have been tested and their aptitudes and skills for the jobs in question have been matched against company standard and evaluated. The applicants have been grouped into four categories by their abilities, the grouping and values of each category to the company are shown in the table that follows:

Applicant category	*Category value (Rs PM)* *A*	*B*	*C*	*D*	*E*	*F*	*No. of applicants*
I	1,000	1,100	1,500	1,400	1,400	1,450	54
II	1,200	1,250	1,200	1,350	1,400	1,400	57
III	1,000	1,100	1,200	1,400	1,500	1,600	45
IV	1,500	1,500	1,600	1,400	1,400	1,500	74

How many applicants of each category should the personnel manager hire and for which jobs? *(MBA, Delhi, December, 1986)*

26. The demand and production costs vary from month to month in an industry. The following table contains budgeted information of a firm in this industry on the quantity demanded, the production cost per unit and the production capacity in each of the coming five months.

Month	*:*	*Jan*	*Feb*	*Mar*	*Apr*	*May*
Demand	:	200	250	150	80	120
Production Cost	:	24	27	32	30	34
Capacity	:	250	225	250	200	225

It is known that the production in any month can meet demand in that month or can be held for the future. The holding cost is Rs 5 per unit per month.

Chapter Overview

In every workplace, there are jobs to be done and there are people available to do them. But everyone is not equally efficient at different jobs. Someone may be more efficient than another on the one and less efficient on the other job, while it might be otherwise for someone else. The relative efficiency is reflected in terms of the time taken for, or the cost associated with, performing different jobs by different people. An obvious problem for a manager to handle is to assign jobs to various workers in a manner that they can be done in the most efficient way. Interestingly, such problems can be formulated as linear programming problems or as transportation problems and solved as such, but a method, called Hungarian Assignment Method provides an easy route. It allows a manager to obtain answers to the questions like the following:

- *How to assign the given jobs to some workers on a one-to-one basis when times of performance are given for each combination and it is desired to complete the jobs in the least time?*
- *How to deal with the situations when the number of jobs do not match with the number of job performers; when some job(s) cannot be performed by, or is not be given to, a particular performer; or when a certain job has to be given to a particular individual?*
- *How should the salesmen of a company be assigned to different sales zones so that the total expected sales may be maximised?*
- *Given the order of preferences that different managers have for the different rooms in a hotel on one its floors, what pattern of assignment of rooms to the managers will satisfy their requirements the most?*
- *How to schedule the flights between two cities so that the layover times for the crew can be minimised?*

In fact, the assignment method works for any problems in which one-to-one matching is called for in the light of the given payoffs, where the total payoff is sought to be minimised or maximised.

Only ordinary arithmetical skills are required for this chapter. Also needed for some part of it are familiarity with the summation notation, algebraic inequalities, matrices and their transpose.

evaluated, while a problem involving 8 persons and 8 jobs will require enumeration and evaluation of as many as 8! = 40,320 assignments, and a problem with 10 workers and as many jobs, the number of possible assignments works out to be 36,28,800. Therefore, the method is not suitable for real world situations.

6.3.2 Transportation Method

We have observed earlier that an assignment problem can be represented as a classical transportation problem. As such, the problem is amenable to solution by applying the transportation method discussed in the previous chapter. But there is a significant point to note here. The solution obtained by applying this method would be severely degenerate. This is because the optimality test in the transportation method requires that there must be $m + n - 1$ ($= 2n - 1$, when $m = n$) basic variables. For an assignment problem of the order $n \times n$, however, there would be only n basic variables in the solution because here n assignments are required to be made, Thus, a large number of epsilons may be required to be introduced in the solution in order to proceed with this method.

Example 6.2 Solve the assignment problem given in Example 6.1 using transportation method.

The problem is reproduced in Table 6.4, wherein the initial solution using VAM is contained. There are only 4 occupied cells against the required number of 4 + 4 – 1 = 7, and hence the solution is degenerate.

Table 6.4 *Initial Solution: VAM*

Worker	*Job*				*Supply*
	A	*B*	*C*	*D*	
1	45 [ε]	40 [ε]	51	67 [1]	1
2	57	42 [1]	63	55	1
3	49 [ε]	52	48 [1]	64	1
4	41 [1]	45	60	55	1
Demand	1	1	1	1	4

To remove degeneracy, an ε is placed in three independent cells. The solution is reproduced in Table 6.5 and tested for optimality. With a total cost equal to 67 + 42 + 48 + 41 = 198, this solution is found to be non-optimal. Accordingly, a closed path is drawn, starting with the cell 2-*D*, which has the largest Δ_{ij} value equal to 14.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

(c) If there is no row or column with only a single 'zero' element left, then select a row/column arbitrarily and choose one of the jobs (or persons) and make the assignment. Now cross the remaining zeros in the column and row in respect of which the assignment is made.

(d) Repeat steps (a) through (c) until all assignments are made.

(e) Determine the total cost with reference to the original cost table.

Example 6.3 Solve the assignment problem given in Example 6.1 for optimal solution using HAM. The information is reproduced in Table 6.7.

Table 6.7 *Time Taken (in minutes) by 4 Workers*

Worker	Job			
	A	B	C	D
1	45	40	51	67
2	57	42	63	55
3	49	52	48	64
4	41	45	60	55

The solution to this problem is given here in a step-wise manner.

Step 1 The minimum value of each row is subtracted from all elements in the row. It is shown in the reduced cost table, also called *Opportunity Cost Table,* given in Table 6.8.

Table 6.8 *Reduced Cost Table 1*

Worker	Job			
	A	B	C	D
1	5	0	11	27
2	15	0	21	13
3	1	4	0	16
4	0	4	19	14

Step 2 For each column of this table, the minimum value is subtracted from all the other values.

Obviously, the columns that contain a zero would remain unaffected by this operation. Here only the fourth column values would change. Table 6.9 shows this.

Then the solution to the assignment problem proceeds in the manner discussed earlier. The effect of assigning prohibitive cost to such person-job combinations is that they do not figure in the final solution.

Example 6.5 You are given the information about the cost of performing different jobs by different persons. The job-person marking × indicates that the individual involved cannot perform the particular job. Using this information, state (i) the optimal assignment of jobs, and (ii) the cost of such assignment.

Person	Job				
	J_1	J_2	J_3	J_4	J_5
P_1	27	18	×	20	21
P_2	31	24	21	12	17
P_3	20	17	20	×	16
P_4	22	28	20	16	27

Balancing the problem and assigning a high cost to the pairings $P_1 - J_3$ and $P_3 - J_4$, we have the cost table given in Table 6.15.

Table 6.15 *Cost Table*

Person	Job				
	J_1	J_2	J_3	J_4	J_5
P_1	27	18	M	20	21
P_2	31	24	21	12	17
P_3	20	17	20	M	16
P_4	22	28	20	16	27
P_5 (Dummy)	0	0	0	0	0

Now we can derive the reduced cost table as shown in Table 6.16. Note that the cells with prohibited assignments continue to be shown with the cost element M, since M is defined to be extremely large so that subtraction or addition of a value does not practically affect it. To test optimality, lines are drawn to cover all zeros. Since the number of lines covering all zeros is less than n, we select the lowest uncovered cell, which equals 4. With this value, we can obtain the revised reduced cost table, shown in Table 6.17.

Table 6.16 *Reduced Cost Table 1*

Person	Job				
	J_1	J_2	J_3	J_4	J_5
P_1	9	0	M	2	3
P_2	19	12	9	0	5
P_3	4	1	4	M	0
P_4	6	12	4	0	11
P_5 (Dummy)	0	0	0	0	0

Expected Sales Data

Salesman	District				
	D_1	D_2	D_3	D_4	D_5
S_1	40	46	48	36	48
S_2	48	32	36	29	44
S_3	49	35	41	38	45
S_4	30	46	49	44	44
S_5	37	41	48	43	47

Since it is a maximisation problem, we would first subtract each of the entries in the table from the largest one, which equals 49 here. The resultant data are given in Table 6.21.

Table 6.21 *Opportunity Loss Matrix*

Salesman	District				
	D_1	D_2	D_3	D_4	D_5
S_1	9	3	1	13	1
S_2	1	17	13	20	5
S_3	0	14	8	11	4
S_4	19	3	0	5	5
S_5	12	8	1	6	2

Now we shall proceed as usual.

Step 1 Subtract minimum value in each row from every value in the row. The resulting values are given in Table 6.22.

Table 6.22 *Reduced Cost Table 1*

Salesman	District				
	D_1	D_2	D_3	D_4	D_5
S_1	8	2	0	12	0
S_2	0	16	12	19	4
S_3	0	14	8	11	4
S_4	19	3	0	5	5
S_5	11	7	0	5	1

Example 6.8 Write the dual of the problem given in Example 6.1 and obtain the optimal values of the dual variables.

The dual problem corresponding to the given problem is as follows:

Maximise $G = u_1 + u_2 + u_3 + u_4 + v_1 + v_2 + v_3 + v_4$

Subject to

$$u_1 + v_1 \le 45 \qquad u_3 + v_1 \le 49$$
$$u_1 + v_2 \le 40 \qquad u_3 + v_2 \le 52$$
$$u_1 + v_3 \le 51 \qquad u_3 + v_3 \le 48$$
$$u_1 + v_4 \le 67 \qquad u_3 + v_4 \le 64$$
$$u_2 + v_1 \le 57 \qquad u_4 + v_1 \le 41$$
$$u_2 + v_2 \le 42 \qquad u_4 + v_2 \le 45$$
$$u_2 + v_3 \le 63 \qquad u_4 + v_3 \le 60$$
$$u_2 + v_4 \le 55 \qquad u_4 + v_4 \le 55$$
$$u_i, v_j \ge 0, i = 1, 2, 3, 4; j = 1, 2, 3, 4$$

We can obtain the optimal values of the dual variables, u_i and v_j's, from the optimal solution to the problem given in Table 6.6 and reproduced in Table 6.25.

Table 6.25 *Obtaining Optimal Values of Dual Variables*

Worker	Job				u_i
	A	B	C	D	
1	45 [ε]	40 [1]	51	67	0
2	57	42 [ε]	63	55 [1]	2
3	49 [ε]	52	48 [1]	64	4
4	41 [1]	45	60	55	−4
v_j	45	40	44	53	

The optimal values are: $u_1 = 0$, $u_2 = 2$, $u_3 = 4$, $u_4 = -4$, $v_1 = 45$, $v_2 = 40$, $v_3 = 44$, and $v_4 = 53$. These sum upto 184, which is obviously the same as the optimal value of Z. It may further be observed that they satisfy all the constraints and are not restricted in sign.

Table 6.31 *Reduced Cost Table 3*

Typist	Job P	Q	R	S	T
A	2	1	0	3	0
B	6	3	0	9	2
C	0	0	0	0	2
D	2	3	0	3	2
E	3	0	1	0	3

In Table 6.31 as well, 4 lines can cover all zeros. Accordingly, RCT-4 is obtained drawn up as the revised table. This is given in Table 6.32.

Table 6.32 *Reduced Cost Table 4*

Typist	Job P	Q	R	S	T
A	2	1	2	3	[0]
B	4	1	[0]	7	~~0~~
C	~~0~~	[0]	2	~~0~~	2
D	[0]	1	~~0~~	1	~~0~~
E	3	~~0~~	3	[0]	3

In this case, the minimum number of lines to cover all zeros equals 5, which matches with the order of the matrix. Accordingly, assignments have been made as described below:

Typist	*Job*	*Cost*
A	T	75
B	R	66
C	Q	66
D	P	80
E	S	112
	Total	399

This optimal solution, however, is not unique.

Example 6.11 Welldone Company has taken the third floor of a multi-storeyed building for rent with a view to locate one of their zonal offices. There are five main rooms in this to be assigned to five managers. Each room has its own advantages and disadvantages.

With the number of lines to cover all zeros being equal to the order of the given matrix, assignments can be made as shown in Table 6.39. However, the problem has an alternative optimal solution as well. Both of these are given below:

Alternative 1

Salesman	*Territory*	*Sales*
A	IV	220
B	I	160
C	III	190
D	II	175
	Total	745

Alternative 2

Salesman	*Territory*	*Sales*
A	IV	220
B	III	150
C	II	195
D	I	180
	Total	745

If salesman *B* cannot be assigned to territory III (alternative 2), then alternative 1 above may be adopted without adverse effect on sale.

Example 6.13 A firm produces four products. There are four operators who are capable of producing any of these four products. The firm records 8 hours a day and allows 30 minutes for lunch. The processing time in minutes and the profit for each of the products are given below:

Operator	*Products*			
	A	*B*	*C*	*D*
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs) per unit	8	6	5	4

Find the optimal assignment of products to operators. *(CA, November, 1997)*

An 8-hour working day, with a 30-minutes lunch time allowed, implies that net working time available per day is 7 hours and 30 minutes, that is 450 minutes. The number of units of different products which could be produced by the four operators can be calculated by dividing 450 by the given processing times. With the profit per unit of each product being given, we may calculate the profit resulting from each possible assignment. The profit matrix is given in Table 6.40. The values in this matrix are derived as follows. For example, operator 1 can produce 450/15 = 30 units of product *A* which, at a profit rate of Rs 8 per unit, implies a total profit of Rs 240.

Table 6.40 *Profit Matrix*

Operator	*Product*			
	A	*B*	*C*	*D*
1	240	300	225	300
2	360	450	250	300
3	144	180	150	200
4	240	300	225	180

Table 6.48 *Reduced Cost Table 1*

Flight	*B1*	*B2*	*B3*	*B4*
A1	14	13	[0]	3
A2	13	12	7	[0]
A3	[0]	1	6	1
A4	1	[0]	7	12

With a zero in each column as well, there is no need to perform column reductions. Further, since the number of lines covering all zeros is equal to the order of the given matrix, we can obtain the optimal assignment as shown in the table.

The optimal pairing of flights is given below:

Flight No.	*Flight No.*	*Lay-over Time*	*Crew Based At*
A1	*B3*	6	Hyderabad
A2	*B4*	9	Hyderabad
A3	*B1*	16	Hyderabad
A4	*B2*	9	Delhi
	Total	40	

Example 6.15 A company has just developed a new item for which it proposes to undertake a national television promotional campaign. It has decided to schedule a series of one-minute commercials during peak audience viewing hours of 1 to 5 P. M. To reach the widest possible audience, the company wants to schedule one commercial on each of the networks and to have only one commercial appear during each of the four one-hour time blocks. The exposure ratings for each hour, which represent the number of viewers per Rs 10,000 spent, are given below.

Viewing Hours		*Nework*		
	A	*B*	*C*	*D*
1–2 P.M.	27.1	18.1	11.3	9.5
2–3 P.M.	18.9	15.5	17.1	10.6
3–4 P.M.	19.2	18.5	9.9	7.7
4–5 P.M.	11.5	21.4	16.8	12.8

(a) Which newwork should be scheduled each hour to provide the maximum audience exposure?

(b) How would the schedule change if it is decided not to use network A between 1 and 3 P.M.?

(M Com, Delhi, 2004)

(a) To solve this problem, we first multiply each value in the matrix by 10 to express exposure ratings per Rs 1 lakh. It simplifies the calculation work somewhat. Further, being a maximisation problem, we subtract each value from the largest value to get the opportunity loss matrix. The result of these steps is given in Table 6.49.

2. The cost elements in the matrix of a typical assignment problem indicate the cost of performing the particular jobs by particular personnel.
3. An assignment problem is said to be *balanced* when the number of rows in the given matrix matches with the number of columns.
4. An assignment problem is always solved on the assumption that the assignments are to be done on a one-to-one basis.
5. A job assignment problem is *unbalanced* when each of the given workers cannot do each of the given jobs.
6. An assignment problem can be designed and solved as a transportation problem.
7. The transportation method can be used to solve an assignment problem.
8. *Any* unbalanced assignment problem can be balanced by introducing a dummy row or a dummy column.
9. All dummy rows/columns *must* have cost elements equal to zero.
10. Balancing of an unbalanced assignment problem might involve the introduction of dummy row as well as columns.
11. The solution to an assignment problem by transportation method would always be degenerate.
12. The relevant cost element is replaced by a zero in case a certain worker is not to be assigned a particular job.
13. The minimum number of lines required to cover all zeros cannot be more than n, the number of rows/columns.
14. If the minimum number of lines covering all the zeros is smaller than n, then the least uncovered value in the table is added to each value lying at the intersection of lines and subtracted from every other value in the table.
15. A maximising assignment problem can be converted into an equivalent minimisation problem by subtracting each element of the given matrix from a constant, K.
16. In an assignment problem involving assignment of salesmen to different zones to exploit their sales potential fully, if a salesman cannot be assigned to a particular zone, then the relevant cell (involving that salesman and the sales zone) value would be replaced by M for solving the problem.
17. If a constant is subtracted from each element of the cost matrix of an assignment problem, it would not affect the optimal assignment schedule.
18. If all entries in a row of the cost matrix are increased by a constant, it will not affect the optimal solution to the problem.
19. In case multiple zeros are obtained in all rows and columns, multiple optimal solutions are indicated.
20. The variables of the dual problem of a given assignment problem shall be unrestricted in sign.

EXERCISES

1. What is an assignment problem? Is it true to say that it is a special case of the transportation problem? Explain.
2. How can you formulate an assignment problem as a standard linear programming problem? Illustrate.
3. What do you understand by an assignment problem? Give a brief outline for solving it.
4. State and discuss the methods employed for solving an assignment problem.
5. Explain the Hungarian Assignment Method. Is it better than other methods of solving assignment problem? How?
6. How would you deal with the assignment problems where
   (a) some assignments are prohibited?
   (b) the objective function is of maximisation type?

10. A company solicits bids on each of four projects from five contractors. Only one project may be assigned to any contractor. The bids received (in thousands of rupees) are given in the accompanying table. Contractor *D* feels unable to carry out project 3 and, therefore, submits no bid.

Project	Contractor				
	A	*B*	*C*	*D*	*E*
1	18	25	22	26	25
2	26	29	26	27	24
3	28	31	30	–	31
4	26	28	27	26	29

(i) Use the Hungarian method to find the set of assignments with the smallest possible total cost.

(ii) What is the minimum total achievable cost? *(MBA, Delhi, October, 1996)*

11. The following information is available regarding four different jobs to be performed and about the clerks capable of performing the jobs:

Clerks	Jobs (Time taken in hours)			
	A	*B*	*C*	*D*
I	4	7	5	6
II	–	8	7	4
III	3	–	5	3
IV	6	6	4	2

Clerk II cannot be assigned to job A and clerk III cannot be assigned to job B. You are required to find out the optimal assignment schedule and the total time taken to perform the jobs. Also find whether the given problem has more than one optimal assignment schedule. *(MIB, Delhi, 2005)*

12. In the modification of a plant layout of a factory, four new machines M_1, M_2, M_3, and M_4 are to be installed in a machine shop. There are five vacant places *A, B, C, D*, and *E* available. Because of limited space, machine M_2 cannot be placed at *C*, and M_3 cannot be placed at *A*. The cost of locating of machine *i* to place *j*, in rupees, is shown here. Find the optimal assignment schedule.

	A	*B*	*C*	*D*	*E*
M_1	89	91	95	90	91
M_2	92	89	–	90	89
M_3	–	91	94	91	87
M_4	94	88	92	87	88

	Course 1	Course 2	Course 3	Course 4
Prof 1	70	50	70	80
Prof 2	30	70	60	80
Prof 3	30	40	50	70
TA	40	20	40	50

How should he assign his staff to the courses to maximise educational quality in his department?

23. The Captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as follows:

Batsmen	Batting positions				
	I	*II*	*III*	*IV*	*V*
P	40	40	35	25	50
Q	42	30	16	25	27
R	50	48	40	60	50
S	20	19	20	18	25
T	58	60	59	55	53

(i) Find the assignment of batsmen to positions, which would give the maximum number of runs.

(ii) If another batsman *'U'* with the following average runs in batting positions as given below:

Batting position :	I	II	III	IV	V
Average runs :	45	52	38	50	49

is added to the team, should he be included to play in team? If so, who will be replaced by him?

(CA, May, 1992)

24. An agency is booking four groups, each of which would give one performance on a particular day. Because of the local preferences of the people, the agency is expecting different audience sizes for the different groups. Five major cities are under consideration of the agency. Considering the estimates of sales of tickets likely to be made, what group-city combination would you advise?

(Estimated sale of tickets ('000 Rs))

Group	City				
	C_1	C_2	C_3	C_4	C_5
G_1	58	56	21	50	45
G_2	16	34	18	25	15
G_3	39	44	30	64	36
G_4	82	102	71	110	73

25. A company has five plants P_1, P_2, P_3, P_4 and P_5. It now wants to introduce three products *A*, *B* and *C*, but not more than one in a plant. The unit production and distribution costs are shown in tables that follow:

for the decision variables to take only integer values. For example, in production, manufacturing is frequently scheduled in terms of lots, batches or production runs. In allocation of goods, a shipment of goods involves a discrete number of trucks, wagons or aircraft, and fractional values are meaningless. Similarly, situations like sequencing, scheduling and routing decisions necessitate the use of integer programming models. An example is the travelling salesman problem, where it is aimed to determine a least distance route so that the salesman must visit each of a given set of cities, starting and ending his journey from a particular city (may be his home town). Capital budgeting provides another situation calling for the application of integer programming models. Hence, the need for restricting variables to assume only integer values and integer programming. The linear programming problems with variables restricted to non-negative integers are called *pure, all-integer,* or *mixed-integer* problems depending, respectively, on whether all or some of the variables are restricted to integer values. Also, there are '0–1' integer programming problems, which constitute an important class of such problems.

7.2.1 Pure and Mixed Integer Programming Problems

An integer programming problem in which *all* variables are required to be integers is called a *pure*, or *all-integer programming* problem. For example,

Maximise $Z = 20x_1 + 8x_2$

Subject to

$$5x_1 + 7x_2 \le 63$$
$$3x_1 + 5x_2 \le 42$$
$$x_1, x_2 \ge 0 \; x_1, x_2 \text{ integer}$$

is a pure integer programming problem.

On the other hand, an IPP in which only some of the variables are required to be integers, is known as a *mixed* integer programming problem. For example,

Maximise $Z = 20x_1 + 8x_2$

Subject to

$$5x_1 + 7x_2 \le 63$$
$$3x_1 + 5x_2 \le 42$$
$$x_1, x_2 \ge 0 \; x_1 \text{ integer}$$

is a mixed integer programming problem since x_2 is not required to be an integer.

Example 7.1 The owner of a ready-made garments store sells two types of premium shirts, known as Zee-shirts and Star-shirts. He makes a profit of Rs 200 and Rs 300 per shirt on Zee and Star shirts, respectively. He has two tailors, *A* and *B*, at his disposal to stitch the shirts. Tailor *A* can devote a total of 17 hours per day, while tailor *B* can give at the most 15 hours a day. Both type of shirts are stitched by both the tailors. The time needed for stitching a Zee shirt is 2 hours by tailor *A* and 3 hours by tailor *B*. Similarly, a Star shirt requires 4 hours by tailor *A* and 3 hours by tailor *B*. How many shirts of each type should be stitched in order to maximise daily profit. Formulate it as an integer programming problem.

Let the daily output of Zee-shirts and Star-shirts be x_1 and x_2 units, respectively. Considering the profitability of shirts and the availability of the tailor time, the problem may be set up as follows:

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Machine	Set-up Cost (Rs)	Cost per Unit (Rs)	Maximum Production
1	8,000	5	4,000
2	5,000	4	3,000
3	4,000	8	1,000

What is the best production strategy? Only formulate. *(MBA, Delhi, 1994)*

Let x_1, x_2 and x_3 represent the quantities to be produced on machines 1, 2 and 3, respectively, and d_1, d_2 and d_3 indicate whether a machine is to be used (1) or not (0). Accordingly, the fixed cost would be $8{,}000d_1 + 5{,}000d_2 + 4{,}000d_3$, while the variable cost would be $5x_1 + 4x_2 + 8x_3$. The integer programming problem is:

Minimise $\quad Z = 8{,}000d_1 + 5{,}000d_2 + 4{,}000d_3 + 5x_1 + 4x_2 + 8x_3$

Subject to

$$x_1 + x_2 + x_3 \geq 5{,}000$$
$$x_1 \leq 4{,}000d_1$$
$$x_2 \leq 3{,}000d_2$$
$$x_3 \leq 1{,}000d_3$$
$$x_1, x_2 \text{ and } x_3 \geq 0;\ d_1, d_2 \text{ and } d_3 = 0, 1$$

5. The Facility Location Problem Another application of integer programming is facility location problems. Such a problem involves deciding where to locate a limited number of facilities such as plants and warehouses, given a larger number of potential locations. Consider a company engaged in the production of two-wheeler scooter and having a wide range geographical area to serve through its sales outlets. In designing a production and distribution system, the company faces a fundamental problem of determining where to locate its plants in order to strike a balance between fixed costs of operating the plants and transportation costs in the distribution of its scooters to various outlets. On the one extreme, it may decide to have one plant adjacent to each outlet—which does not appear to be worthwhile because while transportation costs would be very low, the fixed operating costs would be very high. The other extreme calls for having only one plant to serve all the sales outlets, which is equally unattractive because it would involve low fixed operating costs but very large transportation costs. The optimal solution to the problem would be somewhere between the two extremes and could be obtained by using integer programming. Consider the following example.

Example 7.6 The Standard Cars Limited currently has four plants for manufacturing and assembling cars and six warehouses where these are transported for subsequent delivery to its dealers/showrooms. The company management has collected the following information:

Annual production capacity of each plant,

Annual demand of cars at various warehouses,

Costs involved in transporting cars from various plants to different warehouses, and

Annual fixed operating costs for each of the plants.

The information is summarised in the following table:

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

(i) The algorithm requires that all the coefficients in the constraints and all the right-handside values of the constraints be integers—it would in fact ensure that when decision variables are integers, the slack and surplus variables should also be integers. A constraint like $3.2x_1 + x_2 \le 15.6$ would have to be replaced as $32x_1 + 10x_2 \le 156$.

(ii) At any stage of working of the algorithm, when two or more constraints have fractional values on the RHS, it is desirable to generate the cut by using the constraint whose RHS has a fractional part closest to 1/2.

7.3.2 Branch and Bound Method

The branch and bound method is another method used to solve integer programming problems. It may be mentioned at the outset that the branch and bound method is not a particular method for solving a particular problem. It refers to a certain procedure for finding an optimal solution and is applied differently in different kinds of problems. This is generally used in what may be called as *combinatorial problems*—problems where there are a finite number of solutions. By applying some rules, these solutions are divided into two parts—one that most probably contains the optimal solution and therefore, should be examined further; and the second part that would not contain the optimal solution and, thus, be left out of further consideration.

Thus, branching and bounding essentially keeps dividing up the feasible region in to smaller and smaller parts until the solution that optimises (maximises or minimises, as the case may be) is determined. Here, we shall demonstrate the use of the method for solving assignment problems (0–1 integer programming problems) and travelling salesman problems.

7.3.3 Solution to Assignment Problems

We have considered in the previous chapter, in solution to assignment problems that since such problems have a finite number of possible solutions, it is possible to solve them by enumeration method. As indicated in this method all the possible solutions to a problem are listed and then the one which optimises is chosen. However, since the number of possible solutions rises very rapidly as people and jobs are added to the problem, it becomes very difficult to handle the problems. Even when computers may be employed, it becomes excessively costly and time-consuming. For example, a problem involving ten workers and ten jobs requires a consideration of 36,28,800 feasible solutions! The branch and bound method also employs enumeration but it does this so efficiently that only a small number of the total number of possible solutions need to be examined individually. The use of this method for solving assignment problems is illustrated with the following example.

Example 7.9 The cost to perform different jobs by different workers is given as follows:

Worker	Job			
	1	2	3	4
A	90	18	48	50
B	72	28	85	80
C	53	92	12	78
D	20	70	70	25

Obtain the optimal assignment of jobs to workers.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Table 7.23 *Reduced Cost Table 3*

City	City 1	2	3	4	5	6
1	*M*	5	0	21	34	19
2	0	*M*	17	9	21	0
3	*M*	8	*M*	0	0	0
4	15	0	4	*M*	4	0
5	24	12	0	0	*M*	*M*
6	20	4	11	7	0	*M*

Assignments : 1–3, 2–1, 3–6, 4–2, 5–4, 6–5
Tour : 1–3–6–5–4–2–1
Length : 90 km.

The result of making the route 3–1 unacceptable is also a tour 1–3–6–5–4–2–1 involving a total distance of 90 km.

Clearly, then, the movement along both the branches of this node results in feasible solutions, but each one of them involves a plan that would make the salesman travel longer than how much he would if he accepts the plan already at hand. To conclude, therefore, the upper bound cannot be reduced any further and the optimal solution to the problem is the tour in the order 1–3–5–6–4–2–1.

The final branch and bound tree to this problem is shown in Figure 7.7.

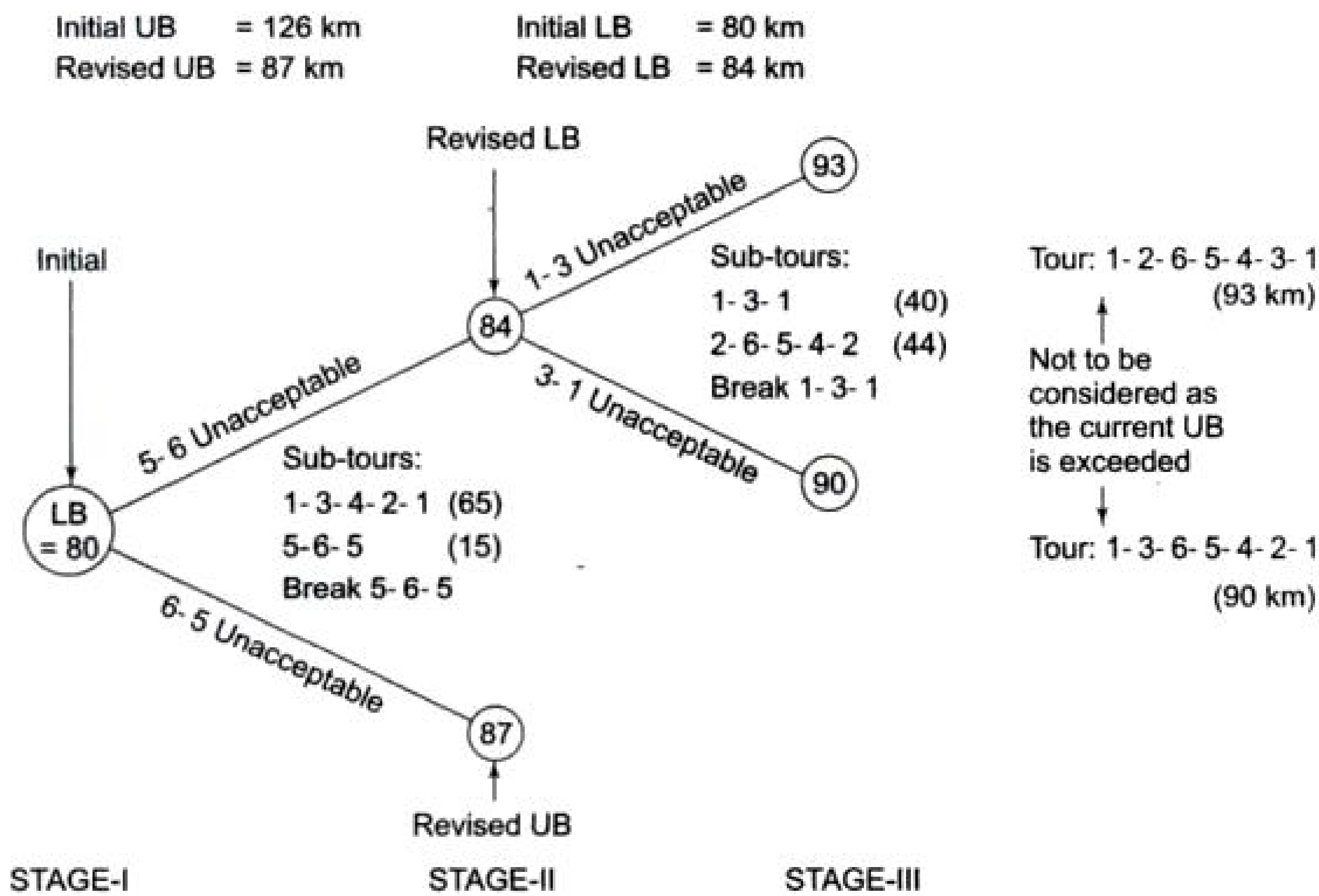


Fig. 7.7 Final Branch and Bound Tree of the Travelling Salesman's Problem

To solve the problem graphically, the raw material constraint is first plotted, as shown in Fig. 7.10, given by line *AT* and the feasible region is shown shaded. Then the constraints involving goals are plotted (by setting deviational variables equal to zero). Since the production capacity constraint may be under- or over-achieved, the feasible area lies on both sides of the line, as indicated by d_1^- and d_1^+. Thus, it does not alter the feasible region. The first goal would, thus, be met and $d_1^- = 0$. Therefore, the focus shifts to the second goal.

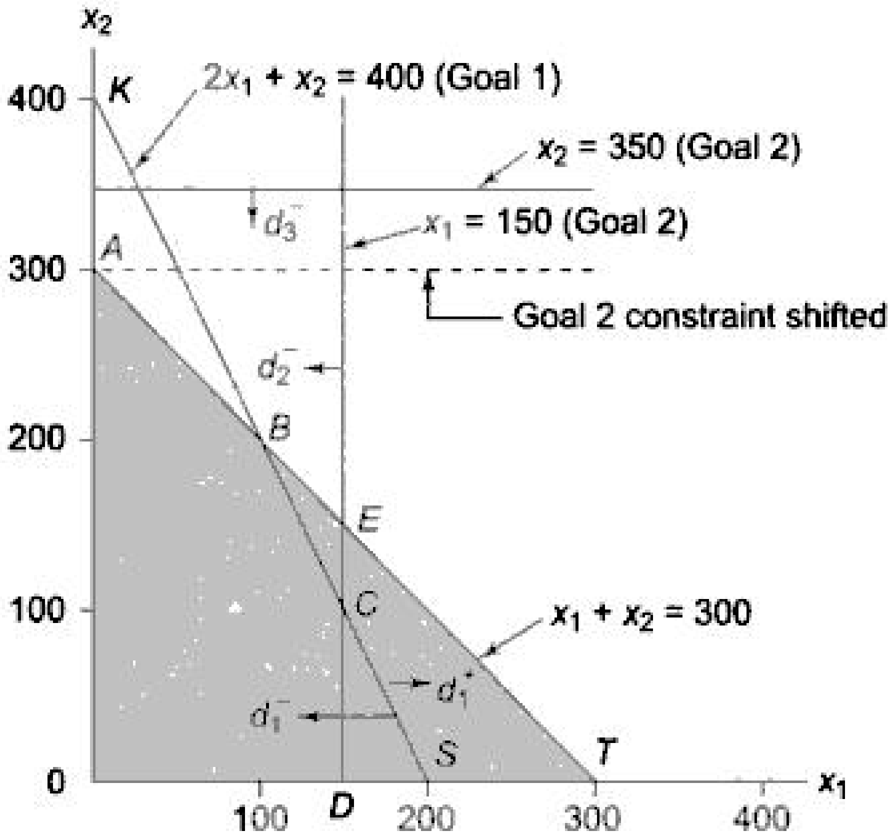


Fig. 7.10 Graphic Solution to GPP: Preemptive

Observing the lines corresponding to the goal constraints, we note that the area DET represents the undesirable deviation and is, therefore, not to be considered. On the other hand, the line depicting the maximum output of the product *B* ($x_2 = 350$) lies beyond the feasible region. Accordingly, the goal constraint line is shown shifted until it touches the boundary of the feasible region at point *A*. Since the product *A* yields a greater profit of Rs 400 over product *B* which yields Rs 300 per unit, a higher penalty is stipulated for deviating from the output target of *A*. Between points *A*, *B*, and *E*, we may obtain the optimal solution by computing the values of d_2^- and d_3^-, and the value of *Z* therefrom.

Point	x_1	x_2	d_2^-	d_3^-	$z = 4d_2^- + 3d_3^-$
A	0	300	150	50	750
B	100	200	50	150	650
E	150	150	0	200	600 ← Minimum

Thus, the optimal solution to the problem is: $x_1 = 150$ and $x_2 = 150$. Substituting these values in the goal 1 constraint equation yields $2 \times 150 + 150 - d_1^+ = 400$ or $d_1^+ = 50$, implying that 50 hours of processing capacity shall be worked overtime. Evidently, since the second-priority goal has not been met fully, there has been no consideration of the next-priority goal.

To continue with the problem, suppose the ranking of the goals is changed so that goals 2 and 3 are interchanged. it would have the effect that while d_1^- would continue to be equal to zero since goal 1 is met, the variable d_1^+ needs to be put equal to zero since a positive value of d_1^+ is an undesirable deviation, in the

Table 7.46 *Simplex Tableau 7: Optimal Solution*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	b_i
x_2	30	0	1	0	–1	0	6	0
x_1	30	1	0	0	1	0	–5	5
S_1	0	0	0	1	4	0	–28	12
S_3	0	0	0	0	0	1	–6	2
c_j		30	30	0	0	0	0	
Solution		5	0	12	0	2	0	$Z = 150$
Δ_j		0	0	0	0	0	–30	

Thus, optimal solution to the problem is : $x_1 = 5, x_2 = 0$ and $Z = 150$.

Example 7.17 A company produces two products, each of which requires stamping, assembly and painting operations. Total productive capacity by operation if it were solely devoted to one product or the other is shown below:

	Productive Capacity (units/week)	
Operation	*Product A*	*Product B*
Stamping	50	75
Assembly	40	80
Painting	90	45

Pro-rata allocation of productive capacity is permissible so also combinations of production of the two products. Demand for the two products is unlimited and the profit on the product A and product B is Rs 150 and Rs 120 respectively. Determine the optimal product-mix. (Non-integer solution for this problem will not be accepted.)

(MBA, Delhi, November 2004)

Let x_1 and x_2 be the number of units of products A and B respectively, produced every week. The IPP is:

Maximise $Z = 150x_1 + 120x_2$ Total profit

Subject to

$$\frac{x_1}{50} + \frac{x_2}{75} \le 1 \quad \text{Stamping}$$

$$\frac{x_1}{40} + \frac{x_2}{80} \le 1 \quad \text{Assembly}$$

$$\frac{x_1}{90} + \frac{x_2}{45} \le 1 \quad \text{Painting}$$

$$x_1, x_2 \ge 0 \text{ and integer}$$

The problem is restated below with constraints expressed in a simplified manner.

Maximise $Z = 150x_1 + 120x_2$

Table 7.52 *Revised Simplex Tableau 3*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	S_5	b_i	b_i/a_{ij}
S_1	0	0	0	1	–1	0	–1/2	0	27/2	–
x_1	150	1	0	0	1	0	–1/2	0	47/2	–
x_2	120	0	1	0	–1	0	1	0	33	33
S_3	0	0	0	0	1	1	–3/2	0	1/2	–
S_5	0	0	0	0	0	0	–1/2*	1	–1/2	1 ←
c_j		150	120	0	0	0	0	0		
Solution		47/2	33	27/2	0	1/2	0	–1/2		
Δ_j		0	0	0	–30	0	–45 ↑	0		

The revised solution is given in Table 7.53

Table 7.53 *Revised Simplex Tableau 4*

Basis		x_1	x_2	S_1	S_2	S_3	S_4	S_5	b_i
S_1	0	0	0	1	–1	0	0	–1	14
x_1	150	1	0	0	1	0	0	–1	24
x_2	120	0	1	0	–1	0	0	2	32
S_3	0	0	0	0	1	1	0	–3	2
S_4	0	0	0	0	0	0	1	–2	1
c_j		150	120	0	0	0	0	0	
Solution		24	32	14	0	2	1	0	Z = 7,440
Δ_j		0	0	0	–30	0	0	– 90	

This solution is found to be optimal for the IPP at hand. Accordingly, $x_1 = 24$, $x_2 = 32$ and Z corresponding to these values being 7,440.

Example 7.18 A salesman must travel from city to city to maintain his accounts. This week he has to leave his home base and visit four other cities and return home. The table shows the distances (in km) between the various cities. The home city is city *A*. Use the assignment method to determine the tour that will minimise the total distances of visiting all cities and returning home.

From City	*To City*				
	A	*B*	*C*	*D*	*E*
A	–	375	600	150	190
B	375	–	300	350	175
C	600	300	–	350	500
D	160	350	350	–	300
E	190	175	500	300	–

(*MBA, Delhi, 2004*)

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

$$4x_1 + 2x_2 \qquad + S_1 = 45 \qquad \text{Deptt. I}$$

$$4x_1 + 4x_2 \qquad + S_2 = 70 \qquad \text{Deptt. II}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, S_1, S_2 \geq 0$$

The solution to this problem, using modified simplex method, is contained in Tables 7.71 through 7.75.

Table 7.71 *Simplex Tableau 1: Non-optimal Solution*

Basis		x_1	x_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	S_1	S_2	b_i	b_i/a_{ij}
d_1^-	P_1	200	300	1	−1	0	0	0	0	0	0	4800	16
d_2^-	P_2	0	1 *	0	0	1	−1	0	0	0	0	15	15 ←
d_3^-	P_3	1	0	0	0	0	0	1	−1	0	0	5	–
S_1	0	4	2	0	0	0	0	0	0	1	0	45	45/2
S_2	0	4	4	0	0	0	0	0	0	0	1	70	35/2
c_j		0	0	P_1	0	P_2	0	P_3	0	0	0		
Δ_j	P_3	−1	0	0	0	0	0	0	1	0	0		
	P_2	0	−1	0	0	0	1	0	0	0	0		
	P_1	−200	−300	0	1	0	0	0	0	0	0		
			↑										

Table 7.72 *Simplex Tableau 2: Non-optimal Solution*

Basis		x_1	x_2	d_1^-	d_1^+	d_2^-	d_2^+	d_3^-	d_3^+	S_1	S_2	b_i	b_i/a_{ij}
d_1^-	P_1	200	0	1	−1	−300	300 *	0	0	0	0	300	1 ←
x_2	0	0	1	0	0	1	−1	0	0	0	0	15	–
d_3^-	P_3	1	0	0	0	0	0	1	−1	0	0	5	–
S_1	0	4	0	0	0	−2	2	0	0	1	0	15	15/2
S_2	0	4	0	0	0	−4	4	0	0	0	1	10	5/2
c_j		0	0	P_1	0	P_2	0	P_3	0	0	0		
Δ_j	P_3	−1	0	0	0	0	0	0	1	0	0		
	P_2	0	0	0	0	1	0	0	0	0	0		
	P_1	−200	0	0	1	300	−300	0	0	0	0		
							↑						

Chapter 8

Sequencing

8.2.1 The Assumptions

There are some general assumptions made to solve the sequencing problems. These are given here:

(a) The processing times on various machines are independent of the order in which different jobs are processed on them.

(b) The time taken by different jobs in going from one machine to another is negligible.

(c) A job once started on a machine would be performed to the point of completion uninterrupted.

(d) A machine cannot process more than one job at a given point of time.

(e) A job would start on a machine as soon as the job and the machine on which it is to be processed are both free.

8.3 SOLUTION TO SEQUENCING PROBLEMS

Sequencing problems can be solved using the *Gantt Chart* and by applying an algorithm. First, we shall consider the Gantt Charts which are used for handling relatively simpler problems involving two machines.

8.3.1 Gantt Charts

We shall illustrate the use of the Gantt Charts with the help of the following examples.

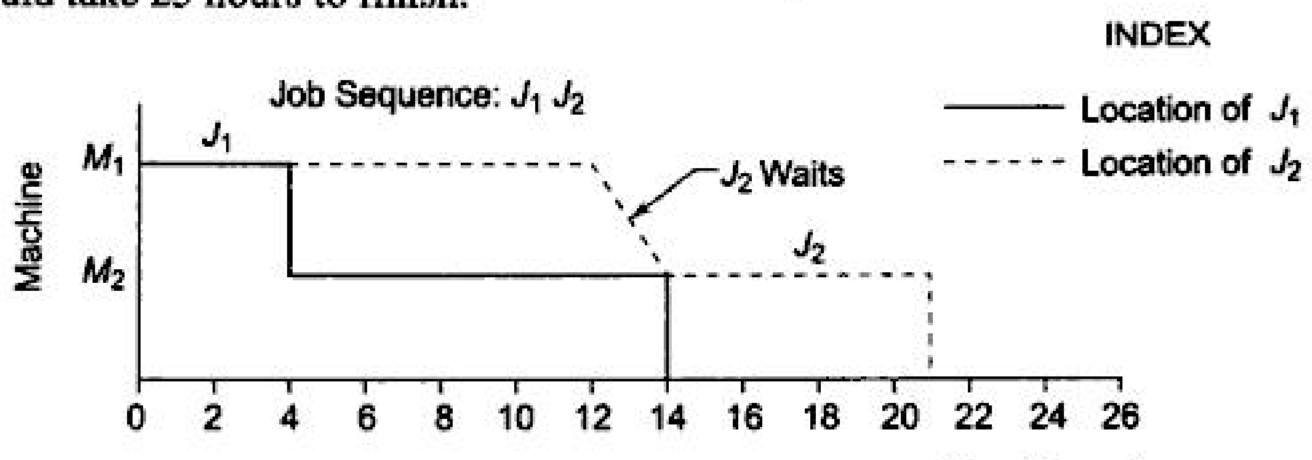
Example 8.1 Suppose that there are two jobs J_1 and J_2, each requiring work on two machines M_1 and M_2, in this order, with the required processing times given as follows:

Processing Time (Hours)

Job	Machine	
	M_1	M_2
J_1	4	10
J_2	8	7

What order of performance of the jobs will involve the least time?

For processing the two jobs, two orders are possible J_1–J_2 and J_2–J_1. Both the alternatives are evaluated in Fig. 8.1. The first part of the figure depicts the job sequences J_1–J_2. According to this sequence, the two jobs can be completed in 21 hours. The other sequence J_2–J_1, is shown in the second part wherein it is clear that the two jobs would take 25 hours to finish.



(a)

Example 8.5 You are given the following data regarding the processing times of some jobs on three machines, *I*, *II* and *III*. The order of processing is *I-II-III*. Determine the sequence that minimises the total elapsed time (*T*) required to complete the jobs. Also evaluate *T* and the idle time of *II* and *III*.

Job	Processing Time (Hours)		
	Machine		
	I	*II*	*III*
A	3	4	6
B	8	3	7
C	7	2	5
D	4	5	11
E	9	1	5
F	8	4	6
G	7	3	12

According to the given information,

$$\text{Min } I_i = 3$$
$$\text{Max } II_i = 5; \quad \text{and}$$
$$\text{Min } III_i = 5$$

Clearly, since Min III_i = Max II_i, the second of the conditions specified is met. Now, we can solve the problem as follows.

First the consolidation table is drawn as shown in Table 8.3.

Table 8.3 *Consolidation Table*

Job	$G_i(= I_i + II_i)$	$H_i(= II_i + III_i)$
A	7	10
B	11	10
C	9	7
D	9	16
E	10	6
F	12	10
G	10	15

According to this, there are two optimal sequences. They are:

S_1 : *A* *D* *G* *B* *F* *C* *E*

S_2 : *A* *D* *G* *F* *B* *C* *E*

We can now evaluate S_1 for the value of *T*. It is done in Table 8.4.

Table 8.4 *Determination of Total Elapsed Time*

Job	Machine I		Machine II		Machine III	
	In	Out	In	Out	In	Out
A	0	3	3	7	7	13
D	3	7	7	12	13	24
G	7	14	14	17	24	36
B	14	22	22	25	36	43
F	22	30	30	34	43	49
C	30	37	37	39	49	54
E	37	46	46	47	54	59

3. Determine the optimal sequencing of jobs from the following data and obtain the value of *T*, the total elapsed time involved:

Job	:	1	2	3	4	5	6	7	8
Machine M_1	:	7	3	6	8	9	5	4	3
Machine M_2	:	8	8	2	4	7	5	6	8

Processing order for all jobs is M_1–M_2.

4. Find the sequence that minimises the total elapsed time required to complete the following tasks on machines M_1 and M_2, in the order M_1, M_2. Also, find the minimum total elapsed time.

Task	:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
M_1	:	2	5	4	9	6	8	7	5	4
M_2	:	6	8	7	4	3	9	3	8	11

(MBA, Chennai, April, 1997)

5. Seven jobs are required to be processed through two machines *A* and *B*. The processing time (in hours) of each jobs on the two machines is given below:

Job	*Processing time*	
	Machine A	*Machine B*
1	10	5
2	20	21
3	5	4
4	25	15
5	15	14
6	12	12
7	6	9

Suggest optimal sequence of processing the jobs and total minimum elapsed time.

6. A company has categorised its investment proposals into seven types. The financial analysts are needed to analyse the risk and return characteristics of these types. Then the proposals are examined by a committee for approval. The time that the analysts and the committee take is based on the size of the project. If it is required to minimise the time taken in evaluation of the seven proposals, how should they be scheduled? The requirements of time are:

Investment	:	*A*	*B*	*C*	*D*	*E*	*F*	*G*
Analysis (by analysts)	:	8	5	10	8	14	10	7
Evaluation (by committee)	:	5	3	7	4	8	6	5

What is the total time needed for evaluation?

7. There are six jobs which must go through two machines *A* and *B* in the order *A*–*B*. Processing time (in hours) is given here:

Job	:	1	2	3	4	5	6
Machine *A*	:	8	10	11	12	16	20
Machine *B*	:	7	15	10	14	13	9

Determine the optimum sequence and the total elapsed time. *(M Com, Delhi, 1987)*

18. Using graphical method, find the minimum elapsed time to perform jobs J_1 and J_2 on machines A through D, the sequence and process timing for which are given as follows:

Job 1	Sequence	:	*A*	*B*	*C*	*D*
	Time (Hrs)	:	2	4	5	1
Job 2	Sequence	:	*B*	*C*	*D*	*A*
	Time (Hrs)	:	6	5	2	3

19. Determine the least time in which the two jobs can be performed on the eight machines. The processing times and the machine sequence for the jobs are given here:

Job 1	Machine sequence	:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	Time (Hours)	:	6	4	5	5	7	2	1	5
Job 2	Machine sequence	:	*C*	*B*	*A*	*D*	*E*	*G*	*F*	*H*
	Time (Hours)	:	5	6	3	4	4	6	1	4

20. Two jobs, A and B, are to be processed on 6 machines. The sequence of machines and the processing times are given in the following table.

Job A	Machine sequence	:	M_1	M_2	M_3	M_4	M_5	M_6
	Time (Hours)	:	6	4	5	3	4	2
Job B	Machine sequence	:	M_1	M_3	M_5	M_4	M_2	M_6
	Time (Hours)	:	4	8	4	3	6	4

What is the minimum time in which both the jobs can be completed?

(d) Decoupling inventories; and

(e) Cycle inventories.

We shall consider these now.

(a) Movement Inventories Movement inventories are also called *transit* or *pipeline inventories*. Their existence owes to the fact that transportation time is involved in transferring substantial amounts of resources. For example, when coal is transported from the coal fields to an industrial town by trains, then the coal, while in the transit, cannot provide any service to the customers for power generation or for burning in furnaces.

(b) Buffer Inventories Buffer inventories are held to protect against the uncertainties of demand and supply. An organisation generally knows the average demand for various items that it needs. However, the actual demand may not exactly match the average and could well exceed it. To meet this kind of a situation, inventories may be held in excess of the average or expected demand. Similarly, the average delivery time (that is, the time elapsing between placing an order and having the goods in stock ready for use, and technically called as the *lead time*) may be known. But unpredictable events could cause the actual delivery time to be more than the average. Thus, excess stocks might be kept in order to meet the demand during the time for which the delivery is delayed. These inventories which are in excess of those necessary just to meet the average demand (during the average lead time period), held for protecting against the fluctuations in demand and lead time are known also by the term *safety stocks*.

The idea of keeping buffer stocks is to render a higher level of customer service and consequently reduce the number of stockouts and back-orders. A stockout occurs when a customer is denied fulfilment of an order because the inventory of the item(s) has runout. In some situations back-ordering is possible (i.e. the order for goods demanded may be fulfilled as soon as the next shipment of the item(s) is received) while in others it is not and the demand might be lost forever leading to temporary/permanent loss of customer goodwill.

It is almost inevitable to keep buffer stocks for the simple reason that perfect prediction of demand and lead time is an exception rather than the rule. We shall discuss later how the optimum level of the buffer stocks to be held can be determined.

(c) Anticipation Inventories Anticipation inventories are held for the reason that a future demand for the product is anticipated. Production of specialised items like crackers well before Diwali, umbrellas and raincoats before rains set in, fans while summers are approaching, or the piling up of inventory stocks when a strike is on the anvil, are all examples of anticipation inventories. The underlying idea is to smoothen the production process for a longer duration on a continuous scale rather than operating with excessive overtime in one period and then letting the system be idle or closing down for the reason of inadequate/no demand for another period.

(d) Decoupling Inventories The idea of the decoupling inventories is to decouple, or disengage, different parts of the production system. As we can observe easily, different machines/equipment and people normally work at different rates—some slower and some faster. A machine, for example, might be producing half the output of the machine on which the item being handled is to be processed the next. Inventories in between the various machines are held in order to disengage the processing on those machines. In the absence of such inventories, different machines and people cannot work simultaneously on a continuous basis. When such inventories are held, then, even if a machine breaks down, the work on others would not stop.

Indeed, if all the machines and people work at the same rate (a rare case, of course) then there would be no need for such inventories. But then, in that case if a machine breaks down, very soon the entire work might come to a standstill.

Because of the first and the second assumptions, there is no need for maintaining a safety stock. Also, because of the first assumption, the inventory decision, as to when to order, is completely specified when the quantity to order is known. Therefore, the basic issue to settle is the determination of the order quantity, Q.

For determining the optimum order quantity, we shall consider two types of costs, viz. the ordering and the holding cost. Since the purchase price of the units is uniform (assumption d), it does not affect the decision as to the quantity of the item to be ordered for purchase and, hence, is irrelevant for the purpose.

According to this, our cost model would, assuming a period of one year, be

$$T(Q) = O(Q) + H(Q)$$

where,

$$Q = \text{the ordering quantity}$$

$$T(Q) = \text{total (variable) annual inventory cost}$$

$$O(Q) = \text{total annual ordering cost}$$

$$H(Q) = \text{total annual holding cost}$$

Let us develop the model with the following example.

Example 9.1 Easton Electronics Co produces 2,000 TV sets in a year for which it needs an equal number of tubes of a certain type. Each tube costs Rs 10 and the cost to hold a tube in stock for a year is Rs 2.40. Besides, the cost of placing an order is Rs 150, which is not related to its size.

Now, if an order for 2,000 tubes is placed, only one order per annum is required. When 1,000 units are ordered for, 2 orders in a year are needed, while if 500 units are ordered to be supplied, then a total of 4 orders per annum are required. Naturally, as the number of orders placed increases the ordering cost goes up. More orders, however, would also imply smaller order quantity and, therefore, decreasing holding costs. Thus, we have a trade-off between the ordering and the holding cost. What we attempt in our EOQ model is, then, to find the order size that minimises the cost function $T(Q)$.

Let us consider for our illustration the computation in a step-wise function.

(a) Total Annual Ordering Cost This is given by the number of times an order is placed, N, multiplied by the ordering (or acquisition) cost per order, A. Thus,

$$O(Q) = N \times A$$

The value of N itself is dependent on the order quantity Q, and the annual demand, D. Here N would be equal to D/Q. Accordingly,

$$O(Q) = \frac{D}{Q} A$$

For our example, when

$N = 1, Q = 2{,}000$ and $O(Q) = 1 \times 150 = \text{Rs } 150,$

$N = 2, Q = 1{,}000$ and $O(Q) = 2 \times 150 = \text{Rs } 300,$

$N = 4, Q = 500$ and $O(Q) = 4 \times 150 = \text{Rs } 600,$

$N = 5, Q = 400$ and $O(Q) = 5 \times 150 = \text{Rs } 750.$

Many a time, the demand is expressed in monetary terms rather than in units. In such cases, if the unit price is known, then the demand may be converted in units by dividing the rupee demand by this unit cost price. Where, however, the unit cost is not given, this cannot be done and we can determine the economic order quantity in rupee terms. When the demand is given in monetary terms, the holding cost must be expressed as a proportion (of the value of the inventory).

Thus, if

cD = the annual demand in rupee terms,

A = the acquisition cost,

i = the holding rate,

we have,

$$\text{EOQ (in rupees)} = \sqrt{\frac{2AcD}{i}}$$

$$\text{Total cost, } T(Q^*) = \sqrt{2AcDi}$$

Obviously, the total cost, $T(Q^*)$ is the same as given in the earlier formulation.

Example 9.2 Using the following data, obtain the EOQ and the total variable cost associated with the policy of ordering quantities of that size.

Annual Demand	= Rs 20,000
Ordering Cost	= Rs 150 per order
Inventory carrying cost	= 24% of average inventory value.

Here

cD = Rs 20,000

A = Rs 150/order

$i = 24\% = 0.24$

$$\text{EOQ (in rupees)} = \sqrt{\frac{2 \times 150 \times 20{,}000}{0.24}}$$

$$= \text{Rs } 5{,}000$$

$$\text{Total cost, } T(Q^*) = \sqrt{2 \times 150 \times 20{,}000 \times 0.24}$$

$$= \text{Rs } 1{,}200$$

A closer look at the example would reveal that it is based on Example 9.1, and, therefore, has the same solution.

Robustness of the EOQ Model An important characteristic of the EOQ model is the robustness which draws from the fact that the total cost curve is relatively flat at the bottom. The model, therefore, tends to give reasonably good results even when parameter values are in error or they vary. We can investigate the sensitivity of the model by considering changes in the ordering quantity and corresponding changes in the total cost. This is useful in some cases as it gives the manager some flexibility in deciding on an order quantity that might better meet the constraints, if any. For example, the EOQ might turn out to be 10.33 crates while the

S below the zero level indicates negative inventory i.e. the number of units backordered. As soon as the lot of Q items is received, the customers whose orders are pending would be supplied their requirements immediately and as such the maximum inventory level would be $Q - S$. The inventory cycle T would be divided in two phases: t_1—the time when inventory is on hand and orders filled as and when they occur, and t_2—when there is a stockout and all the orders are placed on backorder.

In developing the cost function, we would consider the cost of shortages in addition to the holding and the ordering costs. Cost of shortages or the backordering cost is incurred in terms of the labour and special delivery expenses and the loss of customer goodwill (which may be taken to be a function of the time a customer has to wait). Thus,

$$\text{Total (variable) cost} = \text{ordering cost} + \text{holding cost} + \text{shortage cost}$$

Ordering Cost As seen before, if the cost of placing an order be A, and the total demand be D, we have,

$$\text{Annual ordering cost} = \frac{D}{Q} A$$

Holding Cost As noted earlier, t_1 is the period in a given inventory cycle when positive inventory is held. Since the maximum inventory, M, is $Q - S$, the average inventory level equals $(Q - S)/2$. Thus,

$$\text{Holding cost during a given cycle } T = \frac{Q-S}{2} h t_1$$

From the Fig. 9.5, we observe that the quantity $Q - S$ is sufficient to last a period t_1. Thus, $Q - S = t_1 d$, where d is the usage rate. Similarly, a quantity Q is adequate to last a full cycle T, and, therefore, $Q = Td$. Dividing the first of these equations by the second, we get

$$\frac{Q-S}{Q} = \frac{t_1 d}{Td}$$

or

$$t_1 = \frac{T(Q-S)}{Q}$$

Substituting the value of t_1 in this expression for holding cost, we get

$$\text{Holding cost during a given cycle } T = \frac{Q-S}{2} h \frac{T(Q-S)}{Q}$$

$$= \frac{(Q-S)^2 hT}{2Q}$$

There being N orders, and hence N cycles, per year, the annual holding cost would be as follows:

$$\text{Annual holding cost} = \frac{(Q-S)^2 hNT}{2Q}$$

$$= \frac{(Q-S)^2 h}{2Q} \quad \text{(Since } NT = 1 \text{ (year)}$$

Here, fresh supplies are received as soon as the stock level reaches the safety stock level. In this kind of a situation, the average stock held would be exactly equal to $SS + Q/2$. However, a realistic inventory situation would look like the one shown in Fig. 9.8. In this, the demand is not continuous and uniform. Instead, it is discrete and irregular. Displayed in the figure are four inventory cycles. In the first cycle, the demand during lead time (DDLT) is more than expected but there is sufficient safety stock to meet the demand. In the second cycle, the demand is so great that the amount of safety stock cannot meet it. The result is a stockout. The third cycle portrays a condition where the DDLT is less than the expected demand. Lastly, the fourth cycle illustrates a situation wherein the DDLT just matches the expected demand.

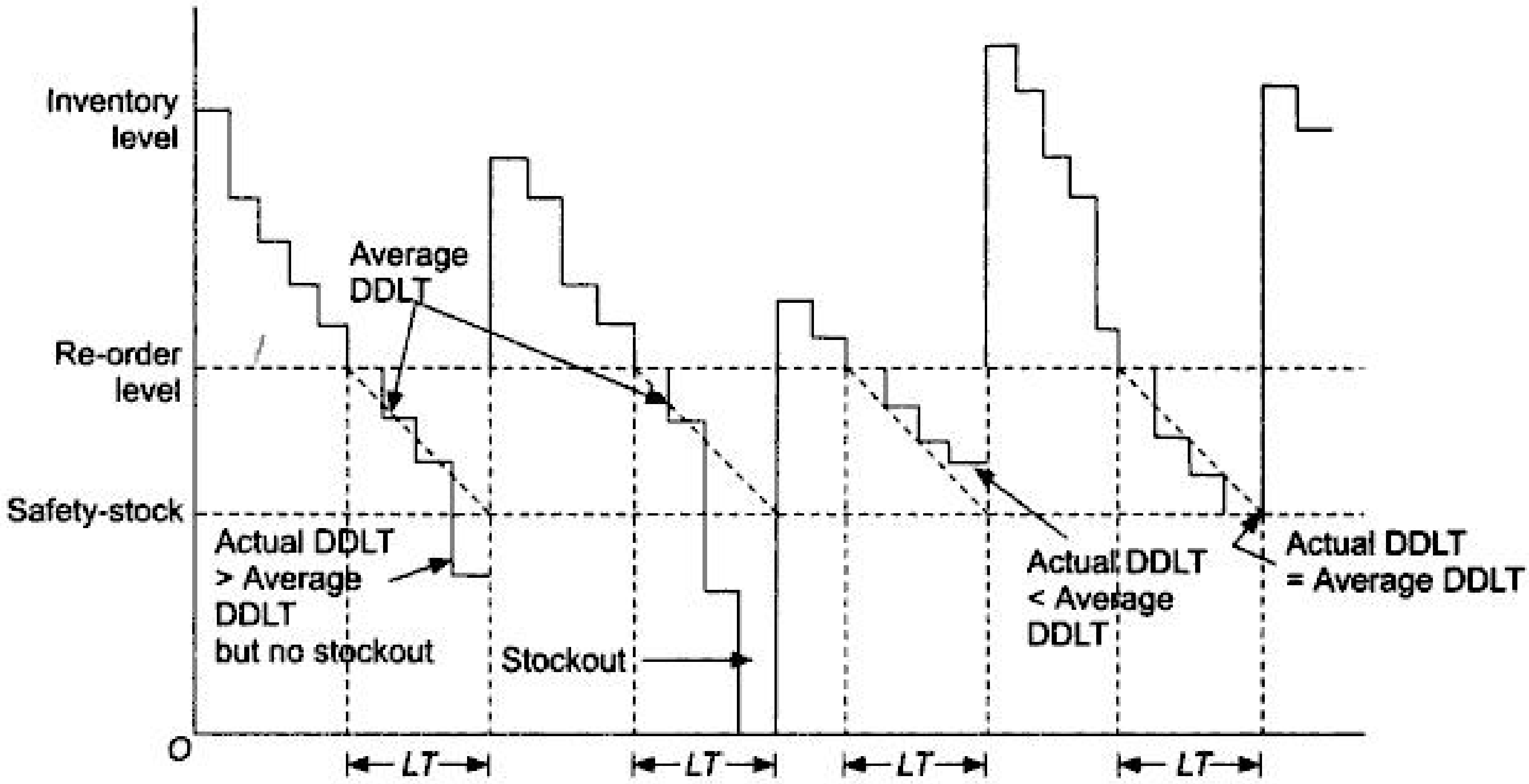


Fig. 9.8 Working of a Real Inventory System

Investment in safety stock is a fairly permanent investment in stock. Even when the demand is irregular and unequal, as illustrated earlier, the amount of (average) stock held would be nearly $Q/2$ plus the safety stock. Notice that the DDLT would be sometimes more, and sometimes less than the expected demand. Therefore, an amount equal to the safety stock would nearly always be carried. The idea of keeping the safety stock is clearly to prevent stockouts and it is the amount of stock that the organisation would always like to preserve for meeting extraordinary situations. As such, investment in safety stocks is made on considerations other than the ones on which the working stock is invested in. In general, higher safety stocks would be called for in situations where costs of stockouts are large; higher level of service (i.e. meeting greater proportion of demand) is sought; significant variations are observed in the lead time and/or lead time demand; and where holding costs are smaller. Naturally, the higher the level of safety stock the greater the service level and, therefore, the smaller the stockouts. The larger safety stock also implies larger holding costs. The management has therefore to strike a balance between the two. The optimal safety stock level is determined where successively declining stockout costs and successively rising holding costs, caused by the successive units added to the safety stock, would balance.

There is no rigid formulation for determining the optimum level of safety stock. The different approaches available for the purpose are based on the demand, the lead time and the stockout costs. The complexity of the situation is determined by the extent and nature of information available about these factors.

Thus, the distribution of demand during the lead time is contained in columns (v) and (vi) of the table and the mean and variance values of this distribution are 12.960 units and 24.53 $(units)^2$, respectively.

Example 9.6 DB plc operates a conventional stock control system based on re-order levels and economic order quantities. The various control levels were set originally based on estimates which did not allow for any uncertainty and this has caused difficulties because, in practice, lead times, demands and other factors do vary.

As part of a review of the system, a typical stock item, part No. X206, has been studied in detail as follows:

Data for part No. X206

Lead Times	*Probability*
15 working days	0.2
20 working days	0.5
25 working days	0.3
Demand per working day	*Probability*
5,000 units	0.5
7,000 units	0.5

Note: It can be assumed that the demands would apply for the whole of the appropriate lead time.

DB plc works for 240 days per year and it costs Re 0.15 p.a. to carry a unit of X 206 in stock. The re-order level for this part is currently 150,000 units and the re-order cost is Rs 1,000.

You are required to

(a) calculate the level of buffer stock implicit in a re-order level of 150,000 units;

(b) calculate the probability of a stockout;

(c) calculate the expected annual stockouts in units;

(d) calculate the stockout cost per unit at which it would be worthwhile raising the re-order level to 175,000 units. *(ICMA, May, 1990, Adapted)*

(a) From the given data, we may obtain the distribution of demand during lead time (DDLT) by multiplying various combinations of the lead time and demand per day with their corresponding probabilities. Finally, the expected DDLT may be calculated. This is done below:

Demand	*Probability*	*Expected Value*
15 × 5,000 = 75,000	0.2 × 0.5 = 0.10	7,500
15 × 7,000 = 105,000	0.2 × 0.5 = 0.10	10,500
20 × 5,000 = 100,000	0.5 × 0.5 = 0.25	25,000
20 × 7,000 = 140,000	0.5 × 0.5 = 0.25	35,000
25 × 5,000 = 125,000	0.3 × 0.5 = 0.15	18,750
25 × 7,000 = 175,000	0.3 × 0.5 = 0.15	26,250
	Total 1.00	123,000

Thus, expected demand during lead time, Exp. DDLT = 123,000 units.

A careful reader will note that this model is not different from the classical EOQ model when the conditions underlying the latter are satisfied. The difference is only of the emphasis—order quantity is emphasized in the EOQ model while the emphasis is on time in the review model.

For situations in which the demand and/or the lead time are *not certain*, safety stock is required to be kept. Under such conditions,

Target Inventory = average review period demand + average lead time demand + safety stock

The safety stock in such a case is determined in the following way. Suppose the demand during review period is normally distributed with mean μ_1 and standard deviation σ_1, and the demand during lead time, DDLT, is normally distributed with mean μ_2 and standard deviation, σ_2; then we first obtain the variance of the demand over the full cycle by combining the two variances as follows:

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

From this, σ can be obtained by extracting its square root. Once this is calculated, the safety stock, *SS*, value can be determined as: $SS = z\sigma$. The value of z, as before, depends upon the service level chosen.

Evidently, the system of periodic reviews generally would require larger provisions for safety stock, than do the fixed order quantity models. This is because, in the fixed order quantity models, there is a continuous monitoring of the stocks and orders are placed as soon as the stock falls to the point of re-order. On the contrary, in the periodic review system, safety stocks must be provided as a protection against the stockouts which may occur during the interval prior to the time point when stocks are to be reviewed, as well as the stockouts that may occur during the lead time. It is amply brought out by the fact that, in a fixed order quantity model, the safety stock is based on the standard deviation of the DDLT alone, while in the periodic review method, the safety stock is based on the standard deviation value which is large in magnitude as it is derived from the combination of the two standard deviation values.

Example 9.9 The Chi-Square Manufacturing Company has adopted the periodic review system for its inventory management. For an item PS-700, the review period is 25 days during which the demand for this item is known to be normally distributed with an average of 1,450 units and standard deviation equal to 150 units, while the lead time demand is also normally distributed with parameters mean = 430 units and standard deviation = 80 units. How much safety stock should be kept in respect of this item when a 90% service level is desired?

From the given information,

$$\mu_1 = 1{,}450,\ \sigma_1 = 150;$$

$$\mu_2 = 430,\ \sigma_2 = 80$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{150^2 + 80^2} = 170 \text{ units}$$

z (corresponding to 90% service level) = 1.28

$\therefore$ Safety Stock = $z\sigma = 1.28 \times 170 = 217.6$ or 218 units (app.)

9.7 Ss SYSTEM

A third inventory management system is called the Ss system, also termed sometimes as *optional replenishment policy*. This system represents a modified periodic review system where a lower limit is placed on the size of the variable re-order quantity. It uses the formulation of the periodic review system as also the re-order point of the fixed order quantity system. In this system, the rule of operation is:

Step 1 Obtain a list of items along with information on their unit cost and the periodic (usually annual) consumption.

Step 2 Determine the annual usage value for each of the items by multiplying unit cost with number of units and rank them in descending order on the basis of their respective usage values.

Step 3 Express the value for each item as a percentage of the aggregate usage value. Now, cumulate the per cent of annual usage values.

Step 4 Obtain the percentage value for each of the items. For n items, each item shall represent $100/n$ per cent. Thus, if there are 20 items involved in classification, then each item would represent $100/20 = 5\%$ of the materials.

Next, cumulate these percentage values as well.

Step 5 Using the data on cumulated values of items and the cumulated percentage usage values, plot the curve by showing these, respectively, on X and Y axes.

Step 6 Determine appropriate divisions for the A, B and C categories. The curve would rise steeply upto a point. This point is marked and the items upto that point constitute the A-type items. Similarly, the curve would only be moderately sloped towards upright. The point beyond which the slope is negligible is marked and the items covered beyond that point are classed as C-type items because they cause only a negligible increase in the cost. The other items are the B-type items for which the curve depicts a gradual upward rise.

After the items are so classified, the inventory decisions are made on the basis of this classification. '*A*' items would call for a strict control and should be delivered near the time of use. The protection against their stockouts may be set as high as 98%. The '*C*' items might be kept in open storage available and as they are demanded they might be issued without formalities. For these, the re-order quantities might be larger than their respective EOQs. Periodic review system may be invoked for the procurement of such items. In regard to the B-class items, the policy would be of a fairly tight control—not as tight as for A-type items of course.

Example 9.11 Perform ABC analysis using the following data:

Item	*Units*	*Unit price (Rs)*	*Item*	*Units*	*Unit price (Rs)*
1	700	5.00	7	6,000	0.20
2	2,400	3.00	8	300	3.50
3	150	10.00	9	30	8.00
4	60	22.00	10	2,900	0.40
5	3,800	1.50	11	1,150	7.10
6	4,000	0.50	12	410	6.20

(d) Re-order level = lead time in days × demand per day

$= 8 \times 9{,}000/300 = 240$ units

(e) No. of days' stock at the re-order level $= 8$ (equal to lead time)

(f) Length of inventory cycle, $T^* = Q^*/D = 2{,}000/9{,}000$

$= 0.222$ year or $0.222 \times 300 = 66.7$ days

Alternatively, T^* (in days) $= Q^*$/demand per day

$= 2{,}000/30 = 66.7$ days

(g) For the present policy of an order quantity $= 3{,}000$ units,

Ordering cost $= 40 \times 3 =$ Rs 120

Holding cost $= (3{,}000/2) \times 0.18 =$ Rs 270

$\therefore$ $T(3{,}000) = 120 + 270 =$ Rs 390

Thus, saving in cost = Rs 390 – Rs 360 = Rs 30 per year.

(h) (i) *Ordering 20% higher than EOQ:*

$$\text{Ordering quantity} = \frac{120}{100} \times 2{,}000 = 2{,}400 \text{ units.}$$

With $Q^* = 2{,}000$ and $Q = 2{,}400$, $k = 2{,}400/2{,}000 = 1.2$

We have
$$\frac{T(Q)}{T(Q^*)} = \frac{1}{2}\left(\frac{1}{k} + k\right) = \frac{1}{2}\left(\frac{1}{1.2} + 1.2\right) = \frac{61}{60}$$

Thus, the cost would increase by 1/60th or $360 \times 1/60 =$ Rs 6.

(ii) *Ordering 40% lower than EOQ:*

In such a situation, $k = 0.60$, and

$$\frac{T(Q)}{T(Q^*)} = \frac{1}{2}\left(\frac{1}{0.60} + 0.60\right) = \frac{17}{15}$$

Thus the increase in cost would be 2/15th over the cost for EOQ, and would equal $360 \times 2/15 =$ Rs 48.

Example 9.13 Yogesh keeps his inventory in special containers. Each container occupies 10 sq ft of store space. Only 5,000 sq ft of the storage space is available. The annual demand for the inventory item is 9,000 containers, priced at Rs 8 per container. The ordering cost is estimated at Rs 40 per order, and the annual carrying costs amount of 25% of the inventory value.

Would you recommend to Yogesh to increase his storage space? If so, how much should be the increase?

(M Com, Delhi, 1990)

Given, $D = 9{,}000$ units, $A =$ Rs 40/order, $c =$ Rs 8/unit and $h = 25\%$ of Rs 8 = Rs 2/unit/year, we have

$$\text{EOQ} = \sqrt{\frac{2DA}{h}}$$

$$= \sqrt{\frac{2 \times 9{,}000 \times 40}{2}} = 600 \text{ units}$$

When item is produced internally:

Cost per unit = 80% of Rs 30 = Rs 24

Set up cost, S = Rs 250 per set up

$$\text{Total cost} = Dc + \frac{D}{ELS} \times S + \frac{ELS}{2} \times \frac{p-d}{p} \times h$$

$$= 2{,}500 \times 24 + \frac{2{,}500}{808} \times 250 + \frac{808}{2} \times \frac{2{,}300}{4{,}800} \times 4$$

$$= \text{Rs } 61{,}548$$

Evidently, the company should manufacture the product internally.

Example 9.19 The annual demand for a component *Z* is 2,08,000 units at a steady weekly rate of 4,000 units. An appropriate formula for calculating the economic batch quantity for production of a component which is being used (at a rate of *s*) and produced (at a rate of *r* per week) at the same time is

$$\text{EBQ} = \sqrt{\frac{2AC_0}{(1 - s/r)\,ip}}$$

The initial cost of installing the line for producing *Z* was Rs 6,000 for a maximum production capacity of 8,000 per week. The operating costs at full capacity are Rs 100 per week for labour, Rs 600 per week for material, Rs 300 per week for variable overhead and Rs 250 per week for fixed overhead. The cost of preparing the production order, producing drawings and so on is Rs 40 each time production is required. Storage costs including interest have been calculated as Rs 2 per unit per annum.

Now,

(a) Calculate the most economic quantity that should be produced each time the line is set up.

(b) Advise the management if it now thinks that there is an opportunity to produce a special one-off order for 50,000 *Z's* for delivery in six months' time. Your answer should consider quantitative and qualitative factors. *(ICMA, May 1983, Adapted)*

(a) The relevant costs are:

Set up cost C_0 = Rs 40 per set up

Storage cost ip = Rs 2 per unit per year

The other costs are not relevant to determine the batch size as they are not related to, and do not affect it. Also, we have:

Annual Demand A = 2,08,000 units

Production rate r = 8,000 units/week

Usage rate s = 4,000 units/week

$$\text{Thus, EBQ} = \sqrt{\frac{2 \times 2{,}08{,}000 \times 40}{\left(1 - \dfrac{4{,}000}{8{,}000}\right) \times 2}} = 4{,}079 \text{ units}$$

In reality, this would be rounded down to 4,000 units, which is the weekly consumption. Thus, the production pattern would be to have a production run each week to produce the weekly requirement.

(b) At present, the variable costs associated with this component are:

Labour	:	Rs 100	
Material	:	Rs 600	
Variable Overhead	:	Rs 300	
		Rs 1,000	per week

From the given information, $Q^* = \sqrt{\dfrac{2 \times 160 \times 20{,}000}{0.20 \times 2}} = 4{,}000$ units

Thus, average stock level $= 700 + 4{,}000/2 = 2{,}700$ units

(c) Ordering cost $= \dfrac{20{,}000}{4{,}000} \times 160 = \text{Rs } 800$

Carrying cost $= 2{,}700 \times 0.40 = \text{Rs } 1{,}080$

$\therefore$ Total cost $= 800 + 1{,}080 = \text{Rs } 1{,}880$

Example 9.24 For a Fixed Order Quantity System, find out the various parameters for an item with the following data:

Annual consumption – 10,000 units, cost of one unit – Re 1, set up cost – Rs 12 per production run, the inventory carrying cost – Re 0.24 per unit. Past lead times: 15 days, 25 days, 12 days, 14 days, 30 days, 17 days.

(ICWA, June, 1985)

From the given information,

(a) EOQ, $Q^* = \sqrt{\dfrac{2AD}{h}} = \sqrt{\dfrac{2 \times 12 \times 10{,}000}{0.24}} = 1{,}000$ units

(b) Determination of safety stock (*SS*):
We know, SS = (max lead time – avg lead time) demand rate
Here max lead time = 30 days
Avg. led time = (15 + 25 + 12 + 14 + 30 + 17)/6 = 18.8 days

(Some people prefer to calculate the average lead time as 'modified average lead time' by excluding the extreme values. Without considering 25 and 30, and considering only 15, 12, 14 and 17 days, the average here may be taken to be 58/4 = 14.5 days. If this is done, the calculations would change accordingly).

Demand rate = 10,000/360 = 27.8 units/day

(This is by assuming 360 working days in a year)

$\therefore$ $SS = (30 - 18.8)\ 27.8 = 312$ units

Now, expected DDLT = average lead time × demand per day
= 18.8 × 27.8 = 523 units

Thus, re-order level, ROL = SS + expected DDLT
= 312 + 523 = 835 units

(c) Average stock $= SS + \dfrac{1}{2}$ EOQ

$= 312 + \dfrac{1}{2} \times 1{,}000 = 812$ units.

Example 9.25 Daily demand for a product *AX*-303 is normally distributed with mean = 60 units and a standard deviation of 6 units. The lead time is constant at 9 days (working). The cost of placing an order is Rs 20 and the annual holding costs are 20% of the unit price of Rs 10. A 95% service level is desired for the customers, who place orders during the re-order period. You are required to determine the order quantity and the re-order level for the item in question, assuming that there are 300 working days during a year.

experienced (so that the demand is for 2,000 units), then the total cost of inventory holding would be 1,000 × 50 = 50,000 and the cost of being out-of-stock would be (2,000 – 1,000) × 100 = Rs 1,00,000. Thus, for this combination, the total cost would be Rs 1,50,000.

The expected costs are obtained by the summation of the products of cost values and their respective probabilities, for each of the strategies. We observe that the optimal policy is not to keep any safety stock.

Table 9.12 *Determination of Optimal Safety Stock Level*

Stockout	*Prob.*	*Safety Stock (No. of units)*					
(No. of units)		*2000*	*1600*	*1000*	*400*	*200*	*0*
2,000	0.01	1,00,000	120,000	1,50,000	1,80,000	1,90,000	2,00,000
1,600	0.02	1,00,000	80,000	1,10,000	1,40,000	1,50,000	1,60,000
1,000	0.03	1,00,000	80,000	50,000	80,000	90,000	1,00,000
400	0.04	1,00,000	80,000	50,000	20,000	30,000	40,000
200	0.10	1,00,000	80,000	50,000	20,000	10,000	20,000
0	0.80	1,00,000	80,000	50,000	20,000	10,000	0
Expected cost		1,00,000	80,400	52,200	25,800	17,800	11,800*

KEY POINTS TO REMEMBER

- Inventories are held for a variety of reasons. Usually, large amounts are invested in them and, therefore, a proper control over them is necessary.
- In context of inventory management, the three questions needed to be addressed to are: (i) How much to order? (ii) When to order? and (iii) How much safety stock be kept?
- The decisions are taken in the context of two inventory management systems, Fixed Order Quantity system and Periodic Review system, having reference to the total cost comprising purchase cost, ordering/set up cost, carrying cost and stockout cost.
- In situations where the demand is given and uniform and there is no variation in the lead time, no safety stock is required to be kept.
- Under the Fixed Order Quantity system, with uniform demand, known lead time and no shortages permitted, the order quantity is determined so as to minimise the cost. For the classical EOQ model, the ordering and carrying costs are the relevant costs and the least-cost order quantity (the economic order quantity, EOQ) is such where they are equal. An order quantity greater than the EOQ saves the ordering cost and increases carrying cost, while an order quantity smaller than EOQ has the opposite effect on these cost elements. In either case, however, the total cost increases. The order level is set where the stock left is equal to demand during lead time.
- A critical assumption of the classical EOQ model is that unit cost is the same whether the order quantity is small or large. The price-break model deals with situations where discount is available for large order quantities. In such cases, the order quantity is established where the total cost including purchase cost, ordering cost and carrying cost is the minimum.

insurance and warehouse charges are *J* % per annum of the inventory value and the price per unit of the item is *P* rupees. What will be the EOQ?

In the above system of inventory, it is proposed to allow stockouts, with a stock out cost of Rs *S* per unit per month. What will be the EOQ and the maximum level of inventory at any time of the year?

(M Com, Delhi, 1999)

13. 'EOQ models, however complex, are restricted by so many assumptions that they have very limited practical value.' Do you agree with this statement? Illustrate your answer.
14. (i The annual demand for an item of inventory is *D* units, ordering costs are *B* rupees per order, transportation costs per order (irrespective of the number of units transported) is *T* rupees, annual interest charges are *x* % of the value of inventory, insurance and warehousing charges amount to *y* % of the value, and the price of the product is *P* rupees per unit. What will be the EOQ?
    (ii) Would the EOQ in the case of gradual receipt of goods ordered be smaller or larger than the EOQ in the case of instantaneous receipt, all other relevant data remaining the same? Why?
    (iii) List the major weaknesses of the classical EOQ model of inventory.
    (iv) List three factors affecting the re-order level of inventory.
15. What is 'safety stock'? Why should it be kept by an organisation?
16. State whether the following statements are correct. Give reasons:
    (i) Safety stock increases as demand increases.
    (ii) In ABC analysis high cost items are most-likely to fall in category *A*, and least cost items are likely to fall in Category *C*.
    (iii) To protect against stockouts, a large batch size is a must.
    (iv) EOQ is based on a balancing between inventory carrying costs and shortage costs.
    (v) Lead time is the time interval elapsing between the placement of a replenishment order and the receipt of last instalment of goods against the order.
17. Discuss the marginal analysis to the determination of the optimal ordering size. In what conditions is it employed?
18. Explain the basics of selective inventory control and state different selection techniques adopted in Inventory Control System. Give a brief note on each. *(ICWA, December, 1983)*
19. (a) "ABC analysis is a very useful approach for selective inventory control but has some major limitations." Do you agree with this statement? Explain how these limitations, if any, can be overcome.
    (b) Explain the concept of the Q-system, the P-system and the Two-Bin system for management of inventories, by giving appropriate examples. *(MBA, Delhi, November, 1999)*
20. Define selective inventory control. Explain the ABC, VED, HML, SDE, S-OS, and FSN bases of inventory classification. Are they mutually exclusive?

Practical Problems

1. A company has determined from its analysis of production and accounting data that, for a part number KC-438, the annual demand is equal to 10,000 units, the cost to purchase the item is Rs 36 per order, and the holding cost is Rs 2/unit/year.
   Determine
   (a) What should the Economic Order Quantity be?
   (b) What is the optimum number of days' supply per optimum order?

Price Schedule:

Order Size	Unit Price
0-9,999	Rs 2.00
10,000-19,999	Rs 1.60
20,000 and above	Rs 1.40

15. The annual demand for an item of inventory is 2400 units. The order processing costs amount to Rs 350 per order. Inventory holding costs are estimated to be 2 percent per month of the value of inventory. The normal price of the product is Rs 10 per unit. However, the supplier offers a quantity discount of 7.5 percent on an order of at least 400 units, and 12.5 percent on an order of 600 units or more. Determine the EOQ. *(M Com, Delhi, 2002)*

16. The Honolulu Company has a contract to supply 5,000 units of an item per year to a dealer. For this item, the company estimates that the ordering cost is Rs 150 every time that an order is made while the carrying cost (p.a.) is reckoned to be 20 percent of the unit price.
The company is negotiating with a dealer who offers to give the following quantity discount.

Order size	Price per unit (Rs)
Less than 1,000	500
1,000-2,999	450
3,000-4,999	400
5,000 or more	350

Recommend to the company the best inventory policy with regard to this item.

17. A manufacturing company of microwave ovens uses Rs 75,000 worth of LED readout circuits annually in its production process. Cost per order is Rs 45, and the carrying charge assessed against this classification of inventory is 25% of the average balance per year. This company follows an EOQ purchasing system and to date has not been offered by discounts on these circuits. Now the supplier has indicated that if the company would buy its circuits four times a year in equal quantities, a discount of 1.5% off list price would be given in return. Would you advise this company to accept this offer? In order to maintain the present total cost, what should be the minimum discount acceptable to the company if four orders of equal sizes are placed in a year? *(CA, November, 1991)*

18. For one of the A-class items the following data are available:

Annual demand = 1,000 units
Ordering cost per order = Rs 400
Inventory carrying cost = 45%
Cost per item = Rs 15

The purchase manager has placed an order for 500 items in the beginning of the year availing 5% discount. At the beginning of the seventh month he procured 250 items without any discount. At the beginning of the eighth month he procured another order for 250 items with a discount of 8%. Had he followed economic order quantity policy what would have been the gain or loss for the organisation? *(ICWA, December, 1995)*

19. (a) In respect of a component costing Rs 10 each, the annual demand is known to be 5,000 units. The cost of placing an order is Rs 200 and the total holding cost is 20% of the average inventory investment.
Determine,
   (i) the most economical order size,
   (ii) the optimal number of orders during a year,
   (iii) the cost of this policy exclusive of the component cost.

Uniformly distributed annual demand for the item	= 24,000 units
Cost of placing an order	= Rs 20
Cost of receiving, inspecting and checking a delivery	= Rs 70
Cost of carrying stock	= Rs 3 per unit per annum
Shortage cost per unit per year	= Rs 2
Lead time	= 1/24th of a year
Interest paid by the company on funds employed	= 12% per annum
Cost of extra storage facilities	= Rs 600 per annum.

34. A dealer supplies you the following information with regard to a product dealt with by him:

Annual demand	= 5,000 units
Buying cost	= Rs 250 per order
Inventory carrying cost	= 30% per year
Price	= Rs 100 per unit

The dealer is considering the possibility of allowing some back-orders to occur for the product. He has estimated that the annual cost of back-ordering (allowing shortage of) the product will be Rs 10 per unit.

(i) What is the optimum number of units of the product that he should buy in one lot?
(ii) What quantity of the product should he allow to be back-ordered?
(iii) How much additional cost will he have to incur on inventory if he does not permit back-ordering?

(M Com, Delhi, 1986)

35. The annual requirement of a commodity is 3,000 units, the cost of placing an order is Rs 300 and the cost of carrying an item in inventory for one year is Rs 20.
(a) Determine the EOQ.
(b) Determine the re-order level if the number of working days in a year is 300 and the lead time is 15 days.
(c) If the maximum lead time is 20 days and the maximum daily demand is 15 units, determine the safety stock required to prevent stockouts.

36. Obtain (i) Economic Order Quantity, (ii) Number of orders, (iii) Re-order level, (iv) Safety stock, for the following inventory problem:

Annual demand	= 36,000 units
Cost per unit	= Re 1
Ordering cost	= Rs 25
Cost of capital	= 15%
Store charge	= 5%
Lead time	= 1/2 month
Safety stock	= 1 month consumption.

37. The daily demand for an electronic machine is approximately 25 items. Every time an order is placed, a fixed cost of Rs 25 is incurred. The daily holding cost per item inventory is Re 0.40. If the lead time is 16 days, determine the economic lot size and re-order point. *(ICWA, December, 1986)*

Model No	*Annual consumption (in pieces)*	*Unit price (in paise)*
501	30,000	10
502	2,80,000	15
503	3,000	10
504	1,10,000	5
505	4,000	5
506	2,20,000	10
507	15,000	5
508	80,000	5
509	60,000	15
510	8,000	10

49. What is selective inventory control? From the following details, draw a plan of ABC selective control.

Item :	1	2	3	4	5	6	7	8	9	10	11	12
Units ('000) :	7	24	1.5	0.6	38	40	60	3	0.3	29	11.5	4.1
Unit cost :	5	3	10	22	1.5	0.5	0.2	3.5	8	0.4	7.1	6.2

(CA, May, 1983)

50. Ten items kept in inventory by the School of Management Studies at State University are listed below. Which items should be classified as '*A*' items, '*B*' items and '*C*' items? What percentage of items is in each class? What percentage of total annual value is in each case?

Item	*Annual usage*	*Value per unit (Rs)*
1	200	40.00
2	100	360.00
3	2,000	0.20
4	400	20.00
5	6,000	0.04
6	1,200	0.80
7	120	100.00
8	2,000	0.70
9	1,000	1.00
10	80	400.00

(CA, May, 1994)

Chapter 10

Queuing Theory

We shall discuss in more details the various elements of a queuing system and then present mathematical results for some specific systems. The elements of a system are:

1. Arrival Process The arrivals from the input population may be classified on different bases as follows: (a) *According to source* The source of customers for a queuing system can be infinite or finite. For example, all people of a city or state (and others) could be the potential customers at a superbazar. The number of people being very large, it can be taken to be infinite. On the other hand, there are many situations in business and industrial conditions where we cannot consider the population to be infinite—it is *finite*. Thus, the ten machines in a factory requiring repairs and maintenance by the maintenance crew would examplify finite population. Removing one machine from a small, finite, population like this will have a noticeable effect on the calls expected to be made (for repairing) by the remaining machines than if there were a large number of machines, say 500.

(b) *According to numbers* The customers may arrive for service individually or in groups. Single arrivals are illustrated by customers visiting a beautician, students reaching at a library counter, and so on. On the other hand, families visiting restaurants, ship discharging cargo at a dock are examples of bulk, or batch, arrivals.

(c) *According to time* Customers may arrive in the system at known (regular or otherwise) times, or they might arrive in a random way. The queuing models wherein customers' arrival times are known with certainty are categorised as *deterministic models* (insofar as this characteristic is concerned) and are easier to handle. On the other hand, a substantial majority of the queuing models are based on the premise that the customers enter the system stochastically, at random points in time.

With random arrivals, the number of customers reaching the system per unit time might be described by a probability distribution. Although the arrivals might follow any pattern, the frequently employed assumption, which adequately supports many real world situations, is that the arrivals are *Poisson* distributed.

2. Service System There are two aspects of a service system—(a) structure of the service system, and (b) the speed of service.

(a) *Structure of the Service System*

By structure of the service system we mean how the service facilities exist. There are several possibilities. For example, there may be

(i) A Single Service Facility A library counter is an example of this. The models that involve a single service facility are called *single server models*. Figure 10.2 (a) illustrates such a model.

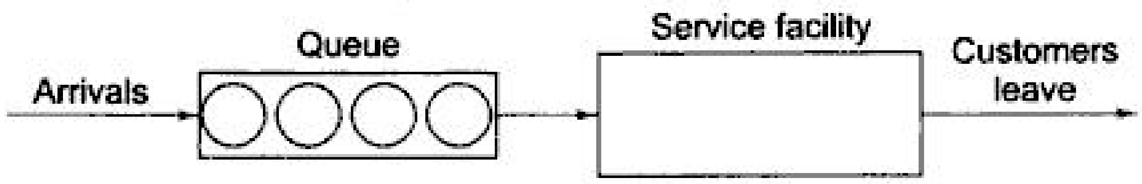


Fig. 10.2(a) Single Server, Single Queue Model

(ii) Multiple, Parallel Facilities with Single Queue That is, there is more than one server. The term parallel implies that each server provides the same type of facility. Booking at a service station that has several mechanics, each handling one vehicle, illustrates this type of model. It is shown in Fig. 10.2 (b).

to serve a customer. Here the arrival and the service rates are each equal to 12 customers per hour. In this situation there shall never be a queue and the banker shall always be busy with work.

Now, suppose that the banker can serve 15 customers per hour. The consequence of this higher service rate would be that the banker would be busy 4/5th of the time and idle in 1/5th of his time. He shall take 4 minutes to serve a customer and wait for 1 minute for the next customer to come. There would be, as before, no queue.

If, on the other hand, the banker can serve only 10 customers per hour, then the result would be that he would be always busy and the queue length will increase continuously without limit with the passage of time. It is easy to visualise that when the service rate is less than the arrival rate, the service facility cannot cope with all the arrivals and eventually the system leads to an explosive situation. The problem in such situations can be resolved by providing additional service station(s). Symbolically, let the arrival rate be λ customers per unit time and the service rate is μ customers per unit time.

Then,

if $\lambda > \mu$ the waiting line shall be formed which will increase indefinitely; the service facility would always be busy; and the service system will eventually fail; and

if $\lambda \leq \mu$ there shall be no waiting time; the proportion of time the service facility would be idle is $1 - \lambda/\mu$.

The ratio $\lambda/\mu = \rho$ is called the *average utilisation*, or the *traffic intensity*, or the *clearing ratio*.

For our present model,

if $\rho > 1$, the system would ultimately fail, and

if $\rho \leq 1$, the system works and ρ is the proportion of time it is busy.

We can easily visualise that the condition of uniform arrival and uniform service rates has a very limited practicability. Such conditions may exist when we are dealing, for example, with movements of items for processing in highly automated plants. However, generally, and more particularly when human beings are involved, the arrivals and servicing time are variable and uncertain. Thus, variable arrival rates and servicing times are the more realistic assumptions. The probabilistic queuing models, as mentioned previously, are based on these assumptions.

10.4.2 Probabilistic Queuing Models

Of the numerous queuing modeis available, we shall consider the following models:

(a) Poisson-exponential, single server model-infinite population;
(b) Poisson-exponential, single server model-finite population; and
(c) Poisson-exponential, multiple server model-infinite population.

In each of these, the words 'Poisson-exponential' indicate that the customer arrivals follow Poisson distribution while the service times are distributed exponentially. To recapitulate, if the arrivals are independent, with the average arrival rate equal to λ per period of time, then, according to the Poisson probability distribution, the probability that n customers will arrive in the system during a given interval T, is given by the following:

$$P\,(n \text{ customers during period } T) = e^{-m}\frac{m^n}{n!}$$

where $m = \lambda T$, and $e = 2.7183$

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Table 10.2 *Calculation of Probabilities of Arrivals*

n	When T = 1/4 hr. P(n)	When T = 1/2 hr. P(n)
0	0.2231	0.0498
1	0.3347	0.1497
2	0.2510	0.2240
3	0.1255	0.2240
4	0.0471	0.1680
5	0.0141	0.1008

(b) Utilisation parameter $\rho = \dfrac{\lambda}{\mu} = \dfrac{6}{10} = 0.6$

(c) Probability that system is idle, $P_0 = 1 - \dfrac{\lambda}{\mu} = 1 - 0.6 = 0.4$

(d) Average time the tailor is free on a 10-hour working day = $P_0 \times$ No. of hours = $0.4 \times 10 = 4$ hours

(e) Probability of n customers in the system, $P_n = \rho^n(1 - \rho)$.

Using this rule, probabilities are given in Table 10.3.

Table 10.3 *Calculation of Probabilities of Customers in System*

n	Probability P(n)
0	0.40
1	0.24
2	0.144
3	0.0864
4	0.05184
5	0.031104

(f) Expected number of customers in the shop,

$$L_s = \frac{\rho}{1-\rho} = \frac{0.6}{1-0.6} = 1.5$$

(g) Expected number of customers in the queue,

$$L_q = \frac{\rho^2}{1-\rho} = \frac{0.6^2}{1-0.6} = 0.36/0.4 = 0.9$$

(h) Expected length of non-empty queues,

$$L_{q'} = \frac{1}{1-\rho} = \frac{1}{1-0.6} = 2.5$$

(i) Expected waiting time in the queue,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{6}{10(10-6)} = \frac{6}{40} \text{ hr.} = 9 \text{ minutes}$$

(j) Expected time a customer spends in the system,

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{10-6} = \frac{1}{4} \text{ hr.} = 15 \text{ minutes}$$

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Rs 70 per hour and will repair machines at an average rate of 7 per hour, while the slow repairman charges Rs 50 per hour and will repair machines at an average rate of 6 per hour. Which repairman should be hired?

For solving this problem, we compare the total expected daily cost for both the repairmen. This would equal the total wages paid plus the cost of non-productive machine hours.

We have, Total wages = Hourly rate × No. of hours

For fast repairman,

Total wages = 70 × 8 (assuming 8-hour shift) = Rs 560

For slow repairman,

Total wages = 50 × 8 = Rs 400

Cost of non-productive time can be calculated as under:

Expected No. of machines in the system × Cost of idle machine hour × No. of hours

Expected number of machines in the system, $L_s = \dfrac{\lambda}{\mu - \lambda}$

Cost of idle machine = Rs 90/hour

No. of hours (assuming 8-hour shift) = 8

For fast repairman,

$\lambda = 4$ machines/hour and $\mu = 7$ machines/hour

$$\therefore \quad L_s = \frac{4}{7-4} = \frac{4}{3} \text{ machines}$$

Thus, cost of non-productive machine time $= \dfrac{4}{3} \times 90 \times 8 = \text{Rs } 960$

For slow repairman,

$\lambda = 4$ machines/hour and $\mu = 6$ machines/hour

$$\therefore \quad L_s = \frac{4}{6-4} = 2 \text{ machines}$$

Accordingly, cost of non-productive machine time = 2 × 90 × 8 = Rs 1,440

The cost of non-productive machine time can be alternatively calculated as follows:

Expected time a machine spends in the system × No. of arrivals per day × Cost of idle time per hour

The product of the first two elements gives the total machine hours lost per day which, when multiplied by the hourly rate, yields the cost of idle (non-productive) time per day.

Now,

$$W_s \text{ (for fast)} = \frac{1}{7-4} = \frac{1}{3} \text{ hour}$$

$$W_s \text{ (for slow)} = \frac{1}{6-4} = \frac{1}{2} \text{ hour}$$

No. of arrivals per day of 8-hours = 8 × 4 = 32

Cost of idle time per hour = Rs 90

(f) Expected number of scooters in the system,

$$L_s = L_q + \frac{\lambda}{\mu} = 2.22 + \frac{2}{0.5} = 6.22 \text{ scooters}$$

(g) Expected waiting time in the queue,

$$W_q = L_q/\lambda = 2.22/2 = 1.11 \text{ hours}$$

(h) Expected time a scooter is in the system,

$$W_s = W_q + \frac{1}{\mu} = 1.11 + \frac{1}{0.5} = 3.11 \text{ hours}$$

Cost analysis Suppose now that the manager of the service station is considering to engage a sixth mechanic in the system. It is given that cost of ill-will created by customer waiting is valued at Rs 12 per customer per hour, while each mechanic is paid Rs 5 per hour. To decide as to whether it would be economical for the manager to add a new mechanic, we shall first work out the operating characteristics of the system with $K = 6$, for the same arrival rate $\lambda = 2$ customers/hour. Then we shall determine the cost implications.

For $K = 6$, $\mu = 1/2$ customer/hour, and $\lambda = 2$ customers/hour, we have $\rho = 2/3 = 0.67$. From Table 10.6, we can obtain the value of P_0 by interpolation. This works out to be 0.001635. The values for other operating characteristics are given here, along with values for $K = 5$ (already worked out):

$$K = 5 : L_s = 6.22 \qquad L_q = 2.22 \qquad W_s = 3.11 \qquad W_q = 1.11$$

$$K = 6 : L_s = 4.558 \qquad L_q = 0.558 \qquad W_s = 2.279 \qquad W_q = 0.279$$

Now we can obtain the cost of the two systems as follows:

System 1: When $K = 5$

Hourly cost of ill-will = Expected number of arrivals per hour × Waiting time is system × Hourly rate

= 2 × 3.11 × 12 = Rs 74.64

Hourly cost of providing service = Number of mechanics × Hourly rate

= 5 × 5 = Rs 25.00

∴ Total cost = 74.64 + 25.00 = Rs 99.64 per hour

System 2: When $K = 6$

Hourly cost of ill-will = 2 × 2.279 × 12 = Rs 54.70

Hourly cost of providing service = 6 × 5 = Rs 30.00

∴ Total cost = 54.70 + 30.00 = Rs 84.70 per hour.

From the results, it is evident that it would be economical to add a sixth mechanic. Similar computations can be done to find the advisability of engaging more mechanics. Calculations will show that adding a seventh mechanic with same efficiency, with arrival rate remaining the same at 2 customers per hour, would raise the average hourly cost to Rs 85.20. Hence, it is not economical to have more than 6 servers in the present case.

$$\text{Total cost per day} = \text{Mechanic's charges} + \text{Cost of waiting}$$
$$= 14 \times 8 + 213.33 = \text{Rs } 325.33$$

With new mechanic:

Here, $\mu = 4$ customers/hour.

Thus, expected waiting time in queue, $W_q = \dfrac{2}{4(4-2)} = \dfrac{1}{4}$ hour

$$\text{Cost of waiting per day} = N \times W_q \times c$$
$$= 16 \times \frac{1}{4} \times 20 = \text{Rs } 80$$
$$\text{Total cost per day} = 18 \times 8 + 80 = \text{Rs } 224$$

Since the total cost is lower with the new mechanic, it is advisable to replace the existing mechanic.

Example 10.14 A warehouse has only one loading dock manned by a three person crew.Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrivals are Poisson distributed. The loading of a truck takes 10 minutes on an average and the loading time can be assumed to be exponentially distributed about this average. The operating cost of a truck is Rs 100 per hour and the members of the loading crew are paid at a rate of Rs 25 per hour. Assuming that the addition of new crew members would reduce the loading time to 7.5 minutes, would you advise the truck owner to add another crew of three persons?

According to given information,

$$\lambda = 4 \text{ trucks/hour},$$
$$\mu = 6 \text{ trucks/hour}.$$

We have,

$$\text{Total hourly cost} = \text{Loading crew cost} + \text{Cost of waiting time}$$

At present

$$\text{Loading crew cost} = \text{No. of loaders} \times \text{Hourly wage rate}$$
$$= 3 \times 25 = \text{Rs } 75 \text{ per hour}$$

Cost of waiting time = Expected waiting time per truck (W_s) × Expected arrivals per hour (λ) × Hourly waiting cost

$$= \frac{1}{6-4} \times 4 \times 100 = \text{Rs } 200 \text{ per hour}$$

Alternatively, cost of waiting time = Expected number of trucks in the system (L_s) × Hourly waiting cost

$$= \frac{4}{6-4} \times 100 = \text{Rs } 200 \text{ per hour}$$

∴ $\quad \text{Total cost} = \text{Rs } 75 + \text{Rs } 200 = \text{Rs } 275 \text{ per hour}$

With proposed crew addition

$$\text{Loading crew cost} = 6 \times 25 = \text{Rs } 150 \text{ per hour}$$

$$\text{Cost of waiting time} = \frac{4}{8-4} \times 100 = \text{Rs } 100 \text{ per hour}$$

∴ $\quad \text{Total cost} = \text{Rs } 150 + \text{Rs } 100 = \text{Rs } 250 \text{ per hour}$

Conclusion: It is advisable to add a crew of three loaders.

Example 10.19 Customers arrive at the rate of twenty per hour and the present serving arrangements can cope with thirty per hour for an eight-hour day.

Using the queuing formulae provided, you are required to calculate and state:

(a) the average time in the queue;

(b) the implied value of customers' time if the owner of the service has considered but rejected a faster service arrangement which would cost an extra Rs 20 for an eight-hour day and would raise the service rate to forty per hour.

The following formulae are given:

$$\text{Average time in system: } \frac{1}{1-\rho} \times \frac{1}{\mu}$$

$$\text{Average time in queue: } \frac{\rho}{1-\rho} \times \frac{1}{\mu}.$$

(ICMA, May, 1983, Adapted)

(a) Here $\lambda = 20\text{/hour}, \mu = 30\text{/hour}$

$$\therefore \rho = 20/30 = 2/3.$$

$$\text{Average time in queue} = \frac{2/3}{1-\frac{2}{3}} \times \frac{1}{30} \text{ hours} = 4 \text{ minutes}$$

(b) With a service rate of 40 per hour, we have = 20/40 = 0.5

$$\text{Previous average time in system} = \frac{1}{1-0.67} \times \frac{1}{30} \times 60 = 6 \text{ minutes}$$

$$\text{New average time in system} = \frac{1}{1-0.5} \times \frac{1}{40} \times 60 = 3 \text{ minutes.}$$

If the break-even value of customers' waiting time is Rs K per hour, then

$$8 \times \left(\frac{6}{60} - \frac{3}{60}\right) \times \lambda \times K < 20$$

or
$$8 \times \frac{3}{60} \times 20 \times K < 20$$

Therefore,
$$K < \frac{60}{24} \quad \text{or} \quad \text{Rs } 2.50 \text{ per hour.}$$

Example 10.20 A bank has two tellers working on savings accounts. The first teller handles withdrawals only while the second teller handles depositors only. It has been found that the service time distributions of both depositors and withdrawals are exponential with a mean service time of 3 minutes per customer. Depositors and withdrawers are found to arrive in a Poisson process throughout the day with mean arrival rate of 16 and 8 per hour respectively. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits? What would be the effect if this could only be accomplished by increasing the service time to 4 minutes?

The existing situation is as follows:

TEST YOUR UNDERSTANDING

Mark the following statements as T (*True*) or F (*False*).

1. In queuing theory, *customers* might include humans, machines, automobiles and so on.
2. A study of queuing theory helps the manager to establish an optimal level of service.
3. Arrival of passengers from aeroplanes at an airport is an example of individual arrivals of customers.
4. Arrival of patients with appointment to a dentist can be described as arrival by Poisson process.
5. The service rate and service time are reciprocals of each other.
6. Multiple servers may be in *series* or in *parallel*.
7. The only way the customers are serviced in queuing situations is the *first-come-first-served* basis.
8. If a waiting customer becomes impatient, it may decide to *renege*.
9. An expectation of a long waiting time, particularly when there are limits on the time and extent of storage capacity available, an arriving customer may *balk*.
10. *Jockeying* is exercised by the customers in shifting to the "fast" moving queues in an attempt to save the waiting time.
11. A queuing model where customer arrivals are at known intervals and the service time is also certain, is a deterministic model.
12. In a deterministic queuing model, the arrival rate must not exceed the service rate, but in a probabilistic model it can.
13. The term *Poisson-exponential* in the context of a queuing model indicates that the arrival rate of customers is Poisson distributed while the service rate is distributed exponentially.
14. If the arrivals occur according to a Poisson distribution, the inter-arrival times would be exponentially distributed.
15. An arrival rate of 10 customers per hour according to Poisson process implies an average inter-arrival time of six minutes.
16. The arrival rate in the Poisson distribution is equal to the mean of the exponential inter-arrival time.
17. In the Poisson-exponential single server model, the system is not workable if the arrival and service rates are equal.
18. In a single server queuing situation, steady state is reached after a sufficiently long period of time if the service rate is greater than the arrival rate.
19. In a Poisson-exponential single server queuing model, the probability of having at least n customers in the system at a random point of time is equal to $\rho^n (1 - \rho)$.
20. For a single server model, $L_{q'} > L_s > L_q$ and $W_s > W_q$.
21. The expected length of system, L_s, should be equal to the expected length of queue *plus* one.
22. It is necessary that the cost of waiting equals cost of servicing at the optimal level of service.
23. In the Poisson-exponential single server model with finite source population, there is a strong dependency relationship between the arrivals.
24. Equal degree of change in the service rate on higher and lower sides produces skewed effect in terms of changes in various parameters.
25. In both single server and multi-server models, *rho* is defined as the ratio of *lambda* to *mu*.
26. The distribution of waiting time is not related to queue discipline used in selecting the waiting customers for service.

(b) Suppose that the two typists are 'pooled'. That is, letters are sent to the two together and are done by whoever is free, in order of arrival. What is the expected waiting time for a letter under this arrangement? *(MBA, Delhi, April, 1996)*

27. An insurance company has three claim adjusters in its Delhi branch office. Claimants against the company are found to arrive in a Poisson fashion, at an average rate of 20 per 10-hour day. The amount of time a server (the adjuster) spends with a customer is found to be exponentially distributed with mean service time of 40 minutes. Claims are processed in the order in which they appear.
    (a) How many hours a week can an adjuster expect to spend with claimants? Assume a 5-day week.
    (b) How much time, on the average, does a claimant spend in the office?
28. Customers arrive at a barber's shop according to Poisson distribution. The average time between successive arrivals is 6 minutes. There are three barbers, all of same efficiency. The service time of the customers is exponentially distributed with a mean equal to 10 minutes per customer. Find (a) the expected number of customers in the shop, (b) the expected time a customer spends in the shop, (c) the average time a customer has to wait in the queue, and (d) the expected number of barbers idle.
29. A post-office has two counters, the first one handles money-orders and registration letters, while the second handles all other business. It has been found that service time distributions for both the counters are exponential with a mean service time of 4 minutes per customer. The customers are found to arrive at the first counter in a Poisson fashion with mean arrival rate of 10 per hour while the customers arrive at the second counter in a Poisson fashion with mean arrival rate of 12 per hour. What would be the effect on the average waiting time for the two types of customers if each counter can handle all types of business?
30. The customers in the Raja Ji Departmental Store, which is being renovated, order their requirements in the different sections, pay cash at the cash counters and then, finally, obtain delivery from the 'delivery counter'. After taking delivery, the customers queue up to leave through a checkout lane.
    The manager is in a fix whether to provide 2 checkout lanes or 3. He needs your help and provides you with the following information.

    "It is estimated that the mean arrival of the customers at the checkout would be 7 per one-minute period and the service rate would be 4 customers per one-minute period per lane. Further, due to goodwill, customers' waiting time costs the store 10 p per minute. The cost to operate a lane is Rs 2.65 per hour."

    For your analysis, assume that (i) customers form a single queue, and (ii) the arrival and service rates are Poisson distributed.
    (a) Show to the manager the comparative values of $P(0)$, L_s, W_q and W_s for 2 and 3 lanes, respectively.
    (b) Suggest to the manager whether to provide 2 lanes or 3.
31. A steel fabrication plant was considering the installation of a second tool crib in the plant to save walking time of the skilled craftsmen who check out equipment at the tool cribs. The Poisson/exponential assumptions about arrivals are justified in this case. The time of the craftsman is valued at Rs 20 per hour. The current facility receives an average of ten calls per hour, with two cribs, each with average five calls per hour. Currently there are two attendants, each of whom services one craftsman at a time, each has a service rate of eight craftsman per hour. Each could do just as well in a separate tool crib. There would be added average inventory costs over the year of Rs 2/hour with the separate tool cribs. However, each craftsman would require six minutes less walking time per call. Evaluate the proposal to set up a new crib so that each attendant would run one crib. *(MBA, Delhi, 2003)*
32. The customers arrive into a large departmental store and wander around the place selecting items of household and other use, and then proceed to checkout stands with a basket of items. There are 4 checkout counters each of which can serve a customer in six minutes, on the average. Assume that the customers arrive at a rate of one every three minutes.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Of the 100 bulbs that would be replaced in the first week, 10 per cent, i.e. 10 bulbs, would fail in the first week of their life, that is to say, week 2; 15 per cent (= 15 bulbs) would fail in week 3, and so on. Similarly, of the total replacements numbering 160 (equal to 150 of the original lot plus 10 out of the 100 replacements in the first week) during the second week, 10 per cent would fail during the third week, 15 per cent during the week 4, ... and so on. The total replacements are shown diagrammatically in the form of *failure tree* in Fig. 11.2.

Starting with the lot of 1,000 bulbs in the beginning, at $t = 0$, 900 bulbs are expected to survive and 100 to fail in week 1. The horizontal line depicts the former and the diagonal line shows the latter. Starting at $t = 1$, 750 of the original bulbs would survive and 150 would fail during the period from $t = 1$ to $t = 2$. Again, the number of surviving bulbs is shown horizontally and the number of those failing is shown diagonally. Of the 100 replacements at $t = 1$, 90 would be surviving at $t = 2$ (shown horizontally) and 10 would fail by $t = 2$ (shown diagonally). In a similar way, horizontal and diagonal lines are drawn for each week.

Next, the sum total of the numbers shown on the diagonals in respect of each week is obtained as written at the bottom. These totals represent the number of replacements in each time period.

Before we see how this information about the number of replacements required every week could be used to take a decision about the optimal interval for replacement, it may be mentioned that it is not necessary to draw a failure tree for obtaining this information. The number of failures expected in different weeks can also be determined algebraically.

In general, if K is the maximum number of periods of life that a unit can have, the number of failures, N_i, in any given period i, can be expressed as follows:

$$N_i = \begin{cases} N_0 \times P_i + \sum_{t=1}^{i-1} N_{i-t}\, p_t & \text{for } i \le K \\ \sum_{t=1}^{K} N_{i-t}\, p_t & \text{for } i > K \end{cases}$$

in which p_t's are the failure probabilities and N_0 is the total number of items. Using the formula, we can easily determine the number of failures in different weeks for our example as given in the Table 11.6.

Table 11.6 *Determination of Number of Failures in Different Weeks*

Week		Expected No. of Failures	
1	$N_1 = N_0 p_1$	$= 1000 \times 0.10$	= 100
2	$N_2 = N_0 p_2 + N_1 p_1$	$= 1000 \times 0.15 + 100 \times 0.10$	= 160
3	$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_1$	$= 1000 \times 0.25 + 100 \times 0.15 + 160 \times 0.10$	= 281
4	$N_4 = N_0 p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1$	$= 1000 \times 0.20 + 100 \times 0.25$	
		$+ 160 \times 0.15 + 281 \times 0.10$	= 277.1
5	$N_5 = N_0 p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1$	$= 1000 \times 0.30 + 100 \times 0.20 + 160 \times 0.25$	
		$+ 281 \times 0.15 + 277.1 \times 0.10$	= 429.8
6	$N_6 = N_5 p_1 + N_4 p_2 + N_3 p_3 + N_2 p_4 + N_1 p_5$	$= 429.8 \times 0.10 + 277.1 \times 0.15 + 281 \times 0.25$	
		$+ 160 \times 0.20 + 100 \times 0.30$	= 216.8
7	$N_7 = N_6 p_1 + N_5 p_2 + N_4 p_3 + N_3 p_4 + N_2 p_5$	$= 216.8 \times 0.10 + 429.8 \times 0.15 + 277.1 \times 0.25$	
		$+ 281 \times 0.20 + 160 \times 0.30$	= 259.6

The determination of the replacement period is shown in Table 11.15.

Table 11.15 *Determination of Optimal Replacement Period*

Year	M_i	P V Factor	P V of M_L	Cum. P V of M_i + Cost	Cum. P V Factor	Average
1	200	1.0000	200.0	8,200	1.0000	8,200
2	600	0.9091	545.5	8,745.5	1.9091	4,581
3	1,000	0.8264	826.4	9,571.9	2.7355	3,499
4	1,400	0.7513	1,051.8	10,623.7	3.4868	3,047
5	1,800	0.6830	1,229.4	11,853.1	4.1698	2,843
6	2,200	0.6209	1,366.0	13,219.1	4.7907	2,759
7	2,600	0.5645	1,467.7	14,686.8	5.3552	2,743*
8	3,000	0.5132	1,539.6	16,226.4	5.8684	2,765
9	3,400	0.4665	1,586.1	17,812.5	6.3349	2,812

Optimal replacement interval for the machine, therefore, is 7 years.

Since minimum average cost for X (= 2,042) is less than the corresponding cost for Y (= 2,743), machine X would be preferable to machine Y.

Example 11.11 The cost of a new car is Rs 10,000. Compare the optimum moment of replacement assuming the following cost informations.

Age of Car (years)	Repair cost per Year	Salvage Value at the End of the Year
1	Rs 5,000	Rs 8,000
2	Rs 10,000	Rs 6,400
3	Rs 10,000	Rs 5,120

Assume that the repairs are made at the end of each year only if the car is to be retained and are not necessary if the car is to be sold for its salvage value. Also assume that the rate of discount is 10%.

(CA, November, 1982)

Here there are three replacement policies to be compared viz. replacement on a yearly, two-yearly and three-yearly basis. For this purpose, we shall consider a six-year period (LCM of 1, 2 and 3) so that each one of the policies would have completed a certain number of cycles. The calculations are shown in Table 11.16.

12.2 PERT/CPM NETWORKS

As mentioned earlier, a project can be viewed as a set of activities or jobs (also called *tasks* or *operations*) that are performed in a certain sequence determined logically or technologically. Therefore, the initial step in PERT/CPM project scheduling process is the determination of all specific activities that comprise the project and their interdependence relationships. Let us take a simple example. Suppose that a new machine is required by a department for which budget approval is needed. It is known that the use of a new machine necessitates employment of an operator who would be trained for operating this machine. The operator can be hired as soon as the proposal of buying the machine is cleared, and trained on a similar machine in the training division of the company. Once the machine is installed and the worker trained, the trial production can commence. In this project, the various activities required to be performed, along with the time needed for their execution, are given in Table 12.1.

Table 12.1 *List of Activities and Precedence Relationships*

Activity	*Description*	*Duration (weeks)*	*Immediate Predecessor(s)*
A	Obtain the budget approval	2	–
B	Obtain the machine	5	*A*
C	Hire the operator	1	*A*
D	Install the machine	1	*B*
E	Train the operator	6	*C*
F	Produce the goods	1	*D, E*

Note that this table contains additional information in the column headed *Immediate Predecessor(s)*. The immediate predecessors for a particular activity are those that must be completed immediately before this may start. For example, from the information contained in the table we can start on activity *A*—obtaining the budget approval—at any time because this is the first activity and it does not depend upon the completion of any prior activities. However, activities *B* and *C* cannot start before *A* is completed, *D* and *E* require, respectively, *B* and *C*, while for *F* to start, *D* and *E* must have been completed.

Once the activities comprising a project, as also the interdependency relationship among them, are clearly identified, they can be portrayed graphically by a *network* or an *arrow diagram*. Because the project planning function begins with a list of all the activities and their precedence relationships, the network can be constructed through the use of a series of *arrows* and *nodes*, thus conveniently expressing the sequential nature of the project.

Each of the activities that make up a project consumes time and resources and has a definable beginning and ending. The arrows, also called *arcs*, in a network represent the various activities of the project. Along with each arrow is given the description and the time estimate of the particular activity it is representing. For example, an activity *A* requiring 3 days is shown represented in Fig. 12.1. The circles at the beginning and at the end of the arrow represent the *nodes*, or the *events*, of beginning and completion, respectively, of the activity in question. The events are points in time and can be considered as milestones of a project. The difference between activities and events should clearly be noted. Whereas an activity is a recognisable part of a project, involving mental or physical work and requiring time and resources for its completion; an event

Example 12.3 Draw an arrow diagram showing the following relationships:

Activity :	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*
Immediate prede. :	–	–	–	*A, B*	*B, C*	*A, B*	*C*	*D, E, F*	*D*	*G*	*G*	*H, J*	*K*	*I, L*

The arrow diagram depicting these relationships is shown in Fig. 12.9.

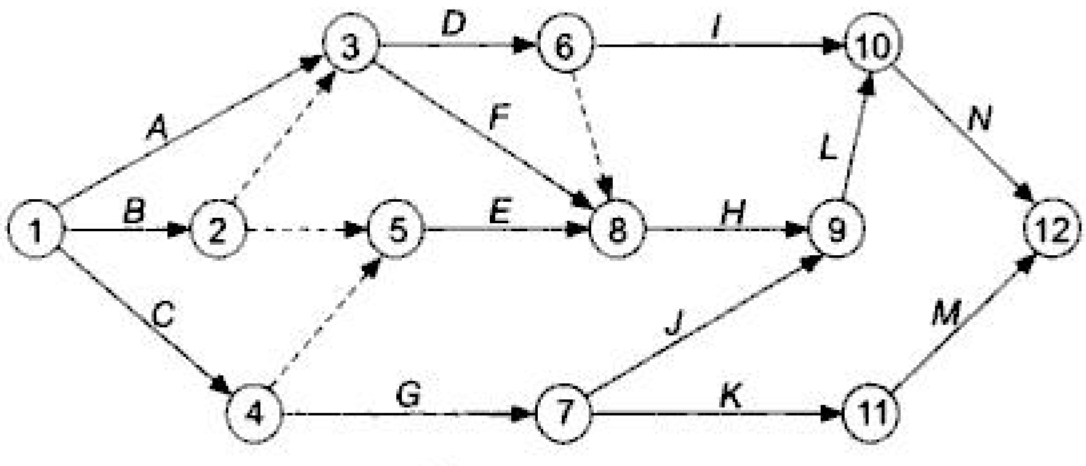


Fig. 12.9 Arrow Diagram

The use of the dummies may carefully be noted here. In particular, the dummy activity 2–5 is necessary here because if it is eliminated, node 5 becomes the ending node of activity *B* and the initial node of activity *E*, implying that *D* and *F* require all *A*, *B*, and *C* to be completed before their start, which is not the case. Inclusion of this activity thus enables us to present the precedence relationships in a correct manner.

12.3 NETWORK ANALYSIS

A project network provides a means to derive a lot of information about the project involved. After the network plan is completed and activity times are known, we analyse it to obtain answers to questions like when the various activities can be scheduled to be performed, how long it will take the project work to be completed, and how much cushion is available for performing the activities in the sense that performance of which of the activities can be delayed and by how much, and which ones cannot be delayed to meet project deadlines. Such analysis is based on the implicit assumption that all resources needed for performing various activities are available in required amounts at the needed times. We consider this analysis first. Later, the question of resources is considered in detail.

12.3.1 Scheduling the Activities: Earliest and Latest Times

After the project network plan is completed and activity times are known, we consider the questions how long the project would take to complete and when the activities may be scheduled. The answers to these questions are provided by an arrow diagram and the time duration of the various activities. These computations involve a forward and a backward pass through the network. The *forward pass* calculations yield the *earliest start* and the *earliest finish* times for each activity, while the *backward pass* calculations render the *latest allowable start* and the *latest finish* times for each activity. We shall demonstrate the calculation of earliest start, earliest finish, latest start and latest finish times of various activities of a project with the help of the following example.

Example 12.4 Information on the activities required for a project is as follows:

start of its successor activity and hence does not interfere in its float. The free float is calculated as the earliest start time for the following activity minus the earliest completion time for this activity.

For activity 2–5, we have free float equal to 8 days because the earliest start of the succeeding activity 5–7 is 13 while the earliest finish time of this activity is 5.

Alternately, free float may be computed as follows. If the slack or float of an *event* is defined as the difference between the earliest and latest event times, we can calculate the slack of the head event and that of the tail event in respect of any activity. In that case,

$$\text{Free float} = \text{Total float} - \text{Head slack}$$

The slack of the event 5 in the network, which is the head event for activity 2–5, is equal to 14 – 13 = 1. Thus, free float for the activity 2–5 is 9 – 1 = 8.

Clearly then, the head event slack indicates the interfering slack, and the free float and the interfering float add up to give the total float. Figure 12.14 shows the free float in respect of the activity 2–5.

Information on the free float of an activity is useful in that it indicates how far the activity in question can be delayed beyond its earliest starting point without affecting the earliest start, and therefore, the total float of the activities following.

Independent Float The independent float time of an activity is the amount of float time which can be used without affecting either the head or the tail events. It represents the amount of float time available for an activity when its preceding activities are completed at their latest and its succeeding activities begin at their earliest time—leaving the *minimum* time available for its performance. Any excess of this minimum time over the duration of the activity is termed as the independent float associated with it. The value of independent float is taken as follows:

$$\text{Independent float} = \begin{matrix}\text{Earliest start time for}\\ \text{the following activity}\end{matrix} - \begin{matrix}\text{Latest finish time for}\\ \text{the preceding activity}\end{matrix} - \begin{matrix}\text{Duration of the}\\ \text{present activity}\end{matrix}$$

Alternatively,

$$\text{Independent float} = \text{Free float} - \text{Tail slack}$$

Thus, the independent float is always either equal to or less than the free float of an activity. Besides, a negative value of the independent float may be obtained. If a negative value is obtained, the independent float is taken to be zero. This is obvious because where the activity duration equals or exceeds the minimum time available for its performance, there will be no independent float.

For the activity 2–5, we have

Earliest starting time for the following activity (5–7) = 13

Latest finishing time for the preceding activity (1–2) = 11

Duration of the activity = 3

Thus, Independent Float = 13 – 11 – 3 = – 1

Similarly, Free float = 8, Tail event slack = 11 – 2 = 9, and Independent float = 8 – 9 = – 1.

Being negative, the independent float is taken to be equal to zero.

Accordingly, we have the normal and minimum duration of the project equal to 20 and 13 days respectively.

Now we shall consider the time-cost relationship for this project when it is crashed successively by a period of one day, to know the total cost of the project for durations of 20 days through 13 days.

The First Crashing In this example, the critical activities are 1–3 and 3–6 for which the cost of reduction per day is Rs 90 and Rs 200. Obviously, we would decide to crash the activity 1–3. Crashing it by a day, the project length is reduced to 19 days and the total cost is equal to Rs 9,830. This is depicted in Table 12.4. For the normal duration of the project it would cost Rs 10,000, equal to the direct cost, the overhead and the penalty cost, which are, respectively, Rs 6,500; Rs 3,200 (= 160 × 20); and Rs 300 (= 100 × 3).

Now we change the duration of the activity 1-3 from 8 to 7 days, as shown in part (b) of the figure. At this stage also, the critical path remains 1–3–6.

The Second, and the Third Crashing For the second crashing, we are faced with the same activities to choose from as in the first crashing, viz. 1–3 and 3–6. The situation is the same in the third crashing. The total project cost equals Rs 9,660 and Rs. 9,490 after the second and the third crashings. Notice that the crashing cost at any given stage is equal to the cumulative cost of crashings till that point.

After the third crashing, the critical paths, each with a length of 17 days, are: 1–3–6; 1–2–4–6; and 1–2–5–6.

The Fourth Crashing To reduce the project length from 17 days to 16, an activity from each of these paths should be chosen. The various alternatives, along with their cost are as follows:

Alternative	Activities	Total Crashing Cost
1	1–3, 1–2	90 + 80 = 170
2	1–3, 4–6, 2–5	90 + 50 + 40 = 180
3	3–6, 1-2	200 + 80 = 280
4	3–6, 4–6, 2–5	200 + 50 + 40 = 290

Thus, we would crash activities 1–3 and 1–2 at a cost of Rs 170. The total cost of the project at this stage is Rs 9,500, and the critical paths, after adjusting the activity timings, are the same as above.

The Fifth Crashing For reducing the length of the project time to 15 days, we have the following alternatives. Notice that the activity 1–3 cannot be crashed any more.

Alternative	Activities	Total Crashing Cost
1	3–6, 1–2	200 + 80 = 280
2	3–6, 4–6, 2–5	200 + 50 + 40 = 290

Now we decide to crash activities 3–6 and 1–2 by a day each, at an additional cost of Rs 280. The project cost now equals Rs 9,620, the critical paths still being 1–3–6; 1–2–4–6; and 1–2–5–6.

The Sixth and the Seventh Crashing At each of these crashings, the only choice open is to crash each of the following activities—one activity from every path at a cost of Rs 290: 3–6, 4–6, and 2–5.

The total cost of the project is Rs 9,750 and Rs 9,880, respectively, after these crashings.

From the table, it is clear that the lowest total cost is Rs 9,490 corresponding to the project duration equal to 17 days, whereas the shortest time to complete the project is 13 days at a total cost of Rs 9,880.

Resources Availability:

No. of operators = 50
Equipment X = 1
Equipment Y = 1
Equipment Z = 1

(CA, November, 1985)

Corresponding to the given information the network is shown in Fig. 12.23. Also shown in the Table 12.5 are the earliest start and the latest allowable start times of the various activities as also their total float values.

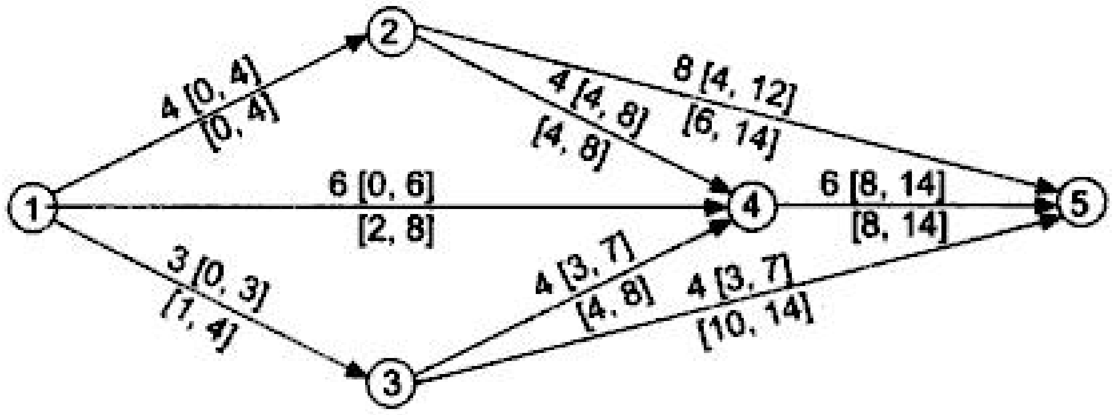


Fig. 12.23 Network

Resource Allocation The first step in the resource allocation procedure is to determine the *ES*, and the *LS* times for each of the activities. In respect of the problem under consideration, these are given in the Table 12.5. Also mentioned are the resource requirements for all the activities.

Table 12.5 *ES and LS Times and Resource Requirements of Activities*

Activity	*Duration*	*ES*	*LS*	*Float*	*Operators*	*Equipment*
1–2	4	0	0	0	30	*X*
1–3	3	0	1	1	20	*Y*
1–4	6	0	2	2	20	*Z*
2–4	4	4	4	0	30	*X*
2–5	8	4	6	2	20	*Z*
3–4	4	3	4	1	20	*Y*
3–5	4	3	10	7	20	*Y*
4–5	6	8	8	0	30	*X*

The resources are allocated by stepping through time, scheduling the various activities as soon as their respective predecessors are scheduled and the resources required for them are available. For this purpose, two sets of activities are defined. The first of these comprises those activities which are eligible for assignment as their predecessors have been scheduled. This may be termed as the *Eligible Activity Set* (EAS). From the eligible activities, those activities are selected, which can start on the particular time when resources are to be allocated, and they are ordered in accordance with the criterion laid earlier. These activities constitute the *Ordered Activities Set* (OAS). Here, it is significant to note that the ordering of the activities on the basis of slack would be the same as obtained on the basis of their late start, *LS*, times. In fact, ordering on the basis of the *LS* values has the advantage that they do not change from one time period to another, whereas slack values would change as the activity that is ready to be scheduled but is not scheduled on a given day. The ordering of the activities in our discussion would be on the basis of their *LS* values.

In our example we start at $T = 0$ (the beginning of the day 1) when the *EAS* comprises activities 1–2, 1–3, and 1–4, since they all can start at the beginning of the project as they do not have any predecessors.

Example 12.11 A project has fourteen activities *A* through *M*. The relationships which obtain among these activities are given here: Construct the network and number them.

A is the first operation

B and *C* can be performed in parallel and are immediate successors to *A*

D, *E*, and *F* follow *B*

G follows *E*

H follows *D*, but it cannot start until *E* is complete

I and *J* succeed *G*

F and *J* precede *K*

H and *I* precede *L*

M succeeds *L* and *K*

The last operation *N* succeeds *M* and *C*. *(ICWA, December, 1985)*

The network corresponding to the given project is shown in Fig. 12.33.

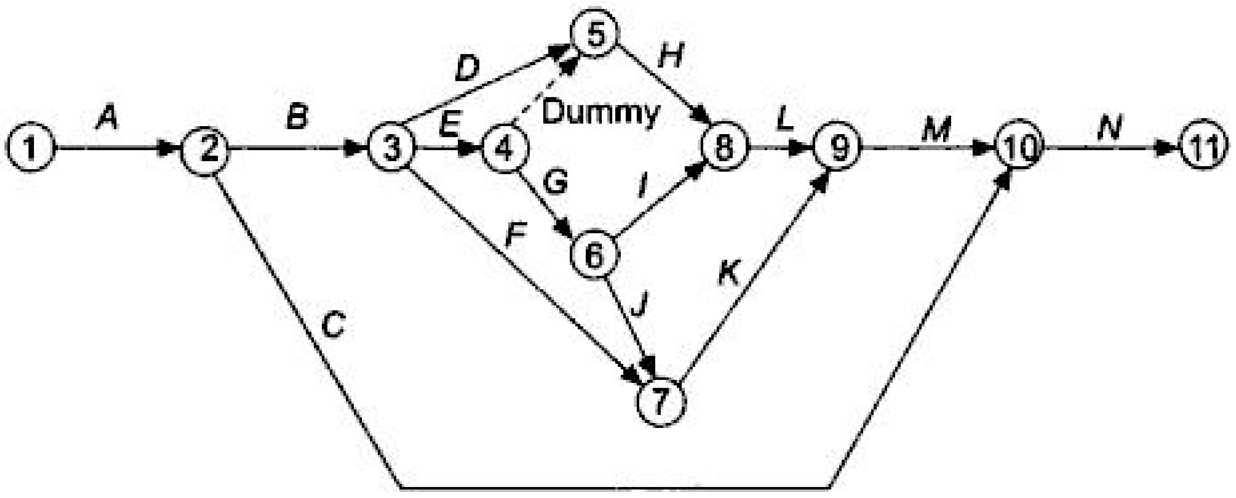


Fig. 12.33 Network Diagram

Example 12.12 Point out the flaws in the network given in Figure 12.34.

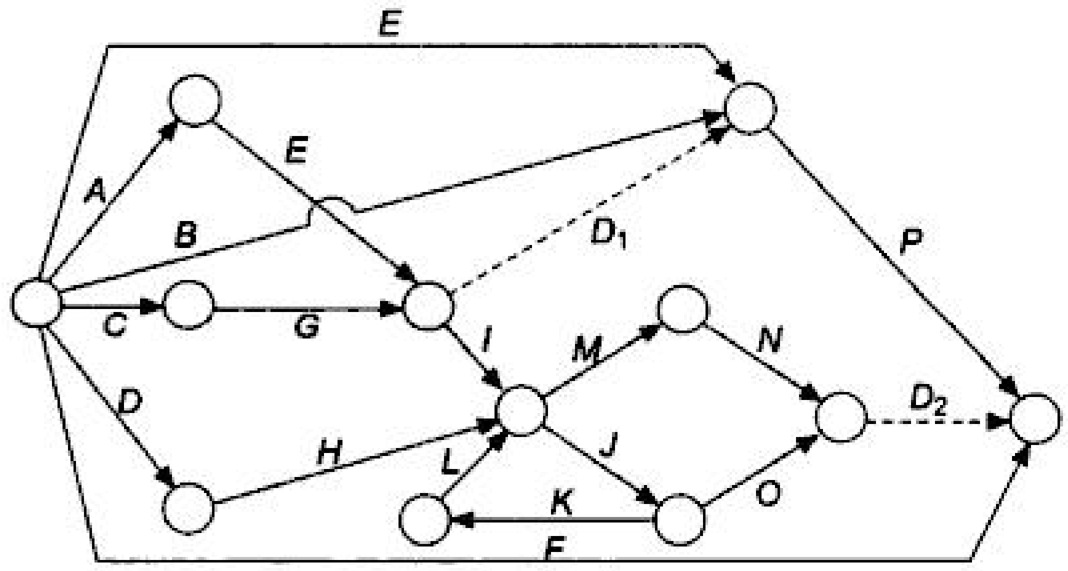


Fig. 12.34 Network Diagram

On observation, we find that

(a) the activities *B* and *E* have the same initial and ending nodes which is not correct,

(d) If activity G(4–6) takes 10 weeks instead of 5, the lengths of the paths (iii) and (iv) would become 44 and 43 weeks respectively. In that case path 1–2–3–4–6–7–9–11 would become critical.

Example 12.16 The following table gives the activities in a construction project and other relevant information:

Activity	Immediate Predecessor(s)	Time (Days)		Direct Cost (Rs)	
		Normal	Crash	Normal	Crash
A	–	4	3	60	90
B	–	6	4	150	250
C	–	2	1	38	60
D	*A*	5	3	150	250
E	*C*	2	2	100	100
F	*A*	7	5	115	175
G	*D, B, E*	4	2	100	240

Indirect costs vary as follows:

Days	:	15	14	13	12	11	10	9	8	7	6
Cost (Rs)	:	600	500	400	250	175	100	75	50	35	25

(a) Draw an arrow diagram for the project.

(b) Determine the project duration which will return in minimum total project cost.

(MBA, Delhi, 1985)

The arrow diagram for the project is given in Fig. 12.38.

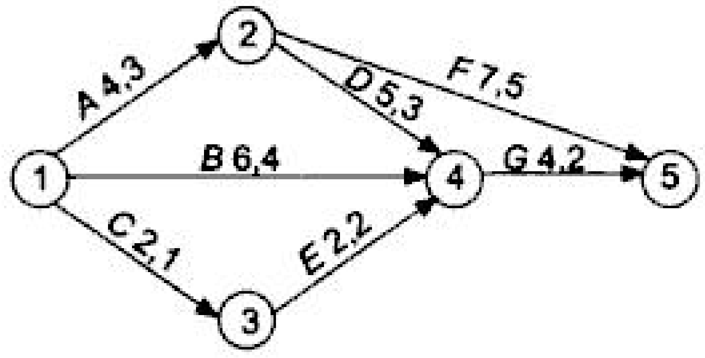


Fig. 12.38 Arrow Diagram

From the diagram we have:

Path	Length	
	Normal Time	Crash Time
1–2–5	11	8* (*Critical*)
1–2–4–5	13* (*Critical*)	8* (*Critical*)
1–4–5	10	6
1–3–4–5	8	5

Thus, the normal duration is 13 days while the minimum completion time of the project is 8 days.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

(b) The contractor stipulates that during the first 26 days only 4 to 5 men and during remaining days 8 to 11 men only can be made available. Rearrange the activities suitably for levelling the manpower resources, satisfying the above condition. *(ICWA, December, 1985)*

The network corresponding to the above project is shown in Fig. 12.40. Also given are the *ES*, *EF*, *LS* and *LF* times of the various activities.

The earliest and the latest start and finishing times are given in Table 12.19, which also contains the information on the total and free float for each of the activities.

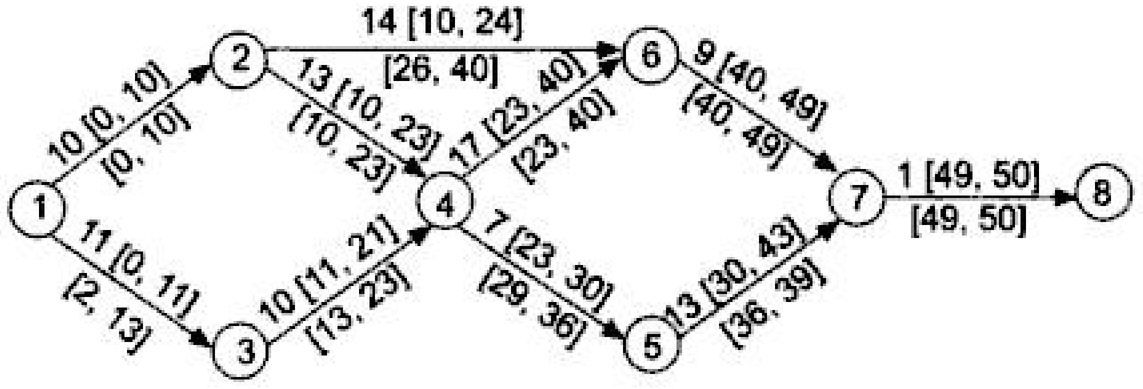


Fig. 12.40 Network Diagram

Table 12.19 *Activity Scheduling Times and Float*

Activity	Duration	ES	EF	LS	LF	Total Float	Free Float
1–2	10	0	10	0	10	0	0
1–3	11	0	11	2	13	2	0
2–4	13	10	23	10	23	0	0
2–6	14	10	24	26	40	16	16
3–4	10	11	21	13	23	2	2
4–5	7	23	30	29	36	6	0
4–6	17	23	40	23	40	0	0
5–7	13	30	43	36	49	6	6
6–7	9	40	49	40	49	0	0
7–8	1	49	50	49	50	0	0

Levelling the Manpower It is sought here to rearrange the activities so that they can be performed with the given resource availabilities, in the stipulated time of 50 days (the project length). The activities 1–2, 2–4, 4–6, 6–7, and 7–8, being critical, would be scheduled first, as shown in Fig. 12.41. Next, activity 1–3 is also scheduled at $t = 0$, followed by the activity 3–4. Although scheduling 1–3 at $t = 0$ would imply that the manpower requirement would rise to 7 on the 11th day of the project, which is in excess of the availability. However, we have no choice and have to schedule it because, otherwise, it would result in a delay of the project.

Since activities 2–6 and 4–5 can both be delayed to $t = 26$ and $t = 29$ respectively, they can both be scheduled at $t = 26$, when more men would be available. Finally, activity 5–7 is scheduled at $t = 33$.

(Contd.)

Activity	*Description of activity*	*Immediate predecessor*	*Expected durations* T_o	T_m	T_P
			(in days)		
J	Train Sales force and Operations Staff	*H, I*	14	17	20
K	Advertise for Suppliers	*C*	1	2	3
L	Check Supplier Samples	*K*	10	15	20
M	Identify key suppliers and define quality standards	*L*	3	5	7
N	Roll-out basic product to a few employees as a method of pre-testing	*M, J*	13	15	17
O	Generate and assess feedback	*N*	20	21	22
P	Modify and make changes wherever necessary	*O*	7	9	14
Q	Update procedures manual	*P*	2	3	4
R	Re-train concerned staff	*Q*	2	2	2
S	Design communications to prospective card-holders	*P*	7	10	13
T	Design advertising schedule	*S*	5	7	9
U	Roll-out final product-LAUNCH	*T, R, G*	4	8	12

where T_o = most optimistic time, T_m = most likely time, and T_p = most pessimistic time

(a) Draw an arrow diagram for the project.

(b) Find the expected project completion time of the project.

(c) Determine the probability of completing the project in 165 days.

(MBA, Delhi, November, 1998)

As a first step, the expected duration, t_e, and variance, σ^2, for each of the activities are calculated and shown below. Here,

$$t_e = \frac{a + 4m + b}{6} \text{ and } \sigma^2 = \left(\frac{b-a}{6}\right)^2$$

where $a = T_o$, $m = T_m$ and $b = T_p$

Activity	*a*	*m*	*b*	t_e	σ^2
A	10	12	14	12	4/9
B	14	15	17	91/6	1/4
C	2	3	4	3	1/9
D	4	6	8	6	4/9
E	10	12	14	12	4/9
F	20	25	27	49/2	49/36
G	10	17	20	49/3	25/9
H	5	6	7	6	1/9
I	7	12	14	23/2	49/36
J	14	17	20	17	1
K	1	2	3	2	1/9

(Contd.)

29. Information about the float times for various activities is essential in effecting resource levelling.
30. It is possible to reduce the normal duration of a project by employing additional resources.
31. The total cost of doing a project is equal to the sum of the direct cost of completing the critical activities and the indirect cost, determined from the time taken to complete the project.
32. While the direct cost is concerned with individual activities of a project, the indirect cost, or overhead, is related to the whole project and is usually a function of time.
33. The *optimal duration* of a project is the minimum time in which it can be completed.
34. In case crashing is possible for a project, the minimum time in which a project can be completed would be given by the length of the critical path determined using crash times for various activities.
35. For an activity, incremental cost can be calculated as follows:

    Incremental cost per day = (normal cost – crash cost)/(normal time – crash time)
36. In crashing, we should always concentrate on critical path and choose between critical activities only.
37. An activity, may or may not be critical, with the minimum crashing cost should be crashed first.
38. When there are multiple critical paths, any of them may be reduced in duration when crashing.
39. It is possible to reduce an activity's duration below its crash time by allocating more resources and funds to it.
40. Resource levelling aims at smoothening of the resource usage rate without changing the project duration.
41. Ordering the activities, which are eligible to be allocated the resources at a given point of time, on the basis of slack is identical to ordering them on the basis of their latest start times.
42. The resource allocation programmes aim to allocate the variable resource supplies to activities of project with a view to complete the project in the minimum time.
43. PERT is probabilistic in nature while CPM is deterministic.
44. Completion time of each activity in a PERT network is assumed to follow beta distribution.
45. The expected completion time of an activity is the weighted average of the optimistic, pessimistic and most-likely times, wherein the "most-likely time" is assigned four times as much weightage as the weightage to the optimistic and the pessimistic times each.
46. The pessimistic time for every activity in a PERT project must always be greater than its optimistic time.
47. In PERT calculations, the critical path is determined by using the optimistic times for various activities.
48. Probability calculations in PERT represent only the probability of completing the activities on the critical path. It is implicitly assumed that the remaining activities would also be over by the time the critical path activities are completed.
49. The standard deviation of the project completion time is obtained by adding the standard deviations of the times of completing the critical activities.
50. In case of more than one critical path for a PERT network, the one with the least variance is selected for making probability calculations.
51. While activity times are variable in PERT, in CPM they are fixed and cannot be varied at all.
52. PERT/Cost aims at cost reduction.
53. PERT/Cost deals with planning, monitoring and controlling project costs.

12. Following are the various activities involved in a project. The cost and time information for these activities is given below:

Activity	Immediately Preceding Activity	Normal Time (days)	Normal Cost (Rs)	Crash Time (days)	Crash Cost (Rs)
A	–	3	140	2	210
B	–	6	215	5	275
C	–	2	160	1	240
D	A	4	130	3	180
E	A	2	170	1	250
F	A	7	165	4	285
G	B, D	4	210	3	290
H	C, E	3	110	2	160

(a) Draw the PERT network.

(b) Find out the critical path and the expected project completion time.

(c) What is the minimum possible project completion time after crashing the activities involved in the project and the associated cost of completing the project? *(MIB, Delhi, 2005)*

13. The following table gives data on normal time and cost, and crash time and cost for a project.

Activity	Duration (Weeks) Normal	Duration (Weeks) Crash	Total Cost (Rs) Normal	Total Cost (Rs) Crash
1–2	3	2	300	450
2–3	3	3	75	75
2–4	5	3	200	300
2–5	4	4	120	120
3–4	4	1	100	190
4–6	3	2	90	130
5–6	3	1	60	110

(i) Draw the network and find out the critical path and the normal project duration.

(ii) Find out the total float associated with each activity.

(iii) If the indirect costs are Rs 100 per week, find out the optimum duration by crashing and the corresponding project costs.

(iv) With the crash duration indicated, what would be the minimum crash duration possible, ignoring indirect costs?

14. In planning a project to introduce a new product in the market, a company lists the various activities, their normal times and costs, and their crash times and costs. They are:

The firm loses Rs 500 for every week beyond a total of 34 weeks. Consider each of the proposals given here individually and state whether it should be accepted by the firm.

(a) Activity *J* can be reduced to 9 weeks by taking resources from activity *K*, thereby raising its expected time to 17 weeks, at no extra cost.

(b) Activity *H* can be reduced to 5 weeks by using overtime, and extra labour at a cost of Rs 1,000.

(c) Activity *G* can be reduced to 7 weeks at a total cost of Rs 500.

(d) Activity *L* can be reduced to 5 weeks by taking resources from activity *K*, thereby increasing its expected time to 16 weeks at no additional cost.

24. Consider the following project. Assume that each activity requires manpower only and that the manpower requirements for the various activities are as follows:

Activity	Duration (Weeks)	Manpower required	Activity	Duration (Weeks)	Manpower required
1–2	2	10	5–7	3	6
2–3	8	18	5–8	5	10
2–4	6	10	6–7	4	8
3–5	12	15	6–9	3	9
3–6	5	6	7–9	4	12
4–5	5	8	8–9	6	8

Further, assume that idle workers are never kept on the payroll, that excess manpower is laid off immediately, and that inadequacies are accommodated by immediate hiring.

(a) Show graphically the level of resource employment through time, assuming that all activities are scheduled as soon as possible.

(b) Show graphically the level of resource employment through time, assuming that non-critical activities are scheduled as late as possible.

25. The following table lists the activities of a maintenance project.

Activity	Duration (in months)	Activity	Duration (in months)
1–2	2	4–7	3
1–3	2	5–8	1
1–4	1	6–8	4
2–5	4	7–9	5
3–6	5	8–9	3
3–7	8		

(i) Draw the project network.

(ii) Find the critical path and duration of the project.

(iii) Suppose we are required to employ a special piece of equipment on activities 1–3, 3–6, 2–5, 5–8 and 8–9, one at a time. Will it affect the duration of the project? Explain.

(CA, November, 1989)

(Contd.)

Task	Precedence	Most Likely	Optimistic	Pessimistic
G Plates for artwork	E	4	3	5
H Designing and printing of jacket	C,D	6	4	9
I Printing and binding of book	F,G	8	6	16
J Inspection and final assembly	I,H	1	1	1

(i) For this PERT network, find the expected task durations and the variances of task durations.

(ii) Draw a network and find the critical path. What is the expected length of the critical path, and what is its variance?

(iii) What is the probability that the length of the critical path does not exceed 32 weeks? 36 weeks?

(MIB, Delhi, 1999)

33. (a) Given that an activity takes generally 20 minutes to perform while the optimistic and pessimistic estimates of its performance are, respectively, one-half and thrice this time, answer the following:

(i) What is the expected duration of this activity?

(ii) What is its variance?

(iii) In scheduling the project, of which this activity is a part, how much time should be provided for this activity?

(b) You are given the following information about a project:

Activity	Duration (days)	Activity	Duration (days)
1 – 2	4	2 – 6	18
1 – 3	7	3 – 5	10
1 – 4	10	3 – 6	16
2 – 3	3	4 – 5	9
2 – 4	8	5 – 6	6
2 – 5	11	5 – 7	11
		6 – 7	8

(i) Draw the network, obtain the scheduling times of various activities, and calculate their total and free slacks.

(ii) Obtain the critical path and find the duration of the project.

(iii) Activities 2 – 6 and 4 – 5 can each be speeded up and reduced by two days at no cost. Do you think it is worth while to reduce these times in terms of the completion time of the project?

(c) Assume that for the project given in (b), the activity durations given represent their expected durations. It is further given that variance along the critical path is 81 days^2. What is the probability of completing the project in 33 days? 44 days?

34. A project consisting of twelve distinct activities is to be analysed by using PERT. The following information is given (time estimates are in days):

(c) Determination of the pay-off function which describes the consequences resulting from the different combinations of the acts and events. The pay-offs may be designated as V_{ij}'s—the pay-off resulting from *i*th event and *j*th strategy.

(d) Choosing from among the various alternatives on the basis of some criterion, which may involve the information given in step (c) only or which may require and incorporate some additional information.

We shall detail the decision analysis in three parts in this chapter. The first part deals with single stage decision making problems, where decisions are taken, by considering the (monetary) paysoffs resulting from various combinations of alternative courses of action and outcomes possible. The second part considers multi-stage decision problems wherein multiple decisions need to be taken one after another. The idea in such cases is to choose the optimal sequence (of decisions) from among the various alternatives. Finally, the third part uses utility, instead of monetary payoffs, as the criterion for decision-making. We consider these in turn now.

13.2 ONE-STAGE DECISION-MAKING PROBLEMS

As indicated, decision-making in case of single stage decision problems calls for (i) identification of the courses of action available to the decision maker in the face of various possible events, (ii) developing a pay-off matrix, and (iii) choosing a particular course of action in accordance with some principle. To understand the decision process under certain conditions, let us consider the following example.

To understand the decision process under uncertain conditions, let us consider the following example.

Example 13.1 A bookstore sells a particular book of tax laws for Rs 100. It purchases the book for Rs 80 per copy. Since some of the tax laws change every year, the copies unsold at the end of a year become outdated and can be disposed off for Rs 30 each. According to past experience, the annual demand for this book is between 18 and 23 copies.

Assuming that the order for this book can be placed only once during the year, the problem before the store's manager is to decide how many copies of the book should be purchased for the next year.

For this problem, since the annual demand varies between 18 and 23 copies, there are six possible events:

E_1	:	18 copies are demanded,	E_4	:	21 copies are demanded,
E_2	:	19 copies are demanded,	E_5	:	22 copies are demanded,
E_3	:	20 copies are demanded,	E_6	:	23 copies are demanded.

Also, there are six possible strategies, or courses of action. They are:

A_1	:	buy 18 copies,	A_4	:	buy 21 copies,
A_2	:	buy 19 copies,	A_5	:	buy 22 copies,
A_3	:	buy 20 copies,	A_6	:	buy 23 copies.

Thus, in this problem there are 6 possible alternatives to choose from, and an equal number of states of nature, or events.

Having listed the possible acts and events, the next step is to construct the pay-off table.

Since the expected regret for strategy A_4 is the least, the optimal policy according to this criterion, as in case of the expected cost principle, is to buy 3 spare parts.

(d) The expected value of perfect information, EVPI, when the pay-off matrix indicates cost, is defined as follows:

EVPI = Expected cost with optimal policy – Expected cost with perfect information

For our example, the expected cost with the optimal policy is 17.4 thousand rupees while the expected cost with perfect information is 9.2 thousand rupees, as follows:

Event	*Cost*	*Prob.*	*Prob.* × *Cost*
E_1 : 0	0	0.1	0.0
E_2 : 1	4	0.2	0.8
E_3 : 2	8	0.3	2.4
E_4 : 3	12	0.2	2.4
E_5 : 4	16	0.1	1.6
E_6 : 5	20	0.1	2.0
		Expected Cost	= 9.2

Thus, EVPI = 17.4 – 9.2 = 8.2 thousand rupees.

12.2.3 Bayesian Decision Rule: Posterior Analysis

In the preceding analysis of decision-making under risk, we have seen how probability information about the various states of nature is used for determining the expected pay-off values resulting from different courses of action. The Bayesian decision rule represents an extension of this. In this approach, the optimal strategy is chosen using the expected value criterion while the expected pay-offs are calculated by using *posterior probabilities*.

In using the Bayesian rule, the preliminary or prior information (from the past experience) of the decision-maker is revised on the basis of some additional information about the states of nature: the prior probabilities are converted into the posterior probabilities using the information. The use of these posterior probabilities for taking the decisions is likely to enable better decisions. In a given situation, the new information may be obtained through test research, raw material sample testing etc.

We shall illustrate the use of this rule by means of the following example.

Example 13.3 Suppose that a firm is in the process of installing a computer system for providing information services, like preparation of pay-rolls, balance sheet etc., to the firms in an industrial area. The firm has to take a decision as to what size of computer system should be purchased. The three alternatives open to the firm are: large-sized, medium-sized and small-sized computer systems. The management of the firm feels that the overall level of acceptance of its services would be either high or low. The pay-offs expected under various event-action combinations together with the estimated probabilities of the likely demand are:

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Continuing with the above situation where $U_{-10,000} = 0$ and $U_{50,000} = 10$, suppose that the decision-maker is asked the question that if he is given a choice between a cash-certain award of Rs 20,000 on the one hand, and bet involving a loss of Rs 10,000 or a gain of Rs 50,000, occurring with equal probabilities on the other, which one would he prefer? If he opts for the first alternative, it is interpreted that according to his preference, the utility of a sum of Rs 20,000 is greater than the expected utility of the bet which equals $0.5\ U_{-10,000} + 0.5\ U_{50,000} = 0.5 \times 0 + 0.5 \times 10 = 5$ units. For him, then $U_{20,000} > 5$ utils. Next, if he is given a choice between a sure amount of Rs 10,000 and the bet as before (that is, the amount – Rs 10,000 and Rs 50,000 with equal probabilities), and if he prefers the second alternative, then it would be similarly concluded that for him $U_{10,000}$ is less than the expected utility of the bet, 5 utils. In a similar way, if we keep on lowering the amount from Rs 20,000, or increasing from Rs 10,000, and asking for the preferences of the decision-maker as before, a value shall be obtained at which he would be indifferent between the two. For him, if this amount equals Rs 16,000, then we say that the utility of Rs 16,000 to that individual equals the expected utility of the given bet. Thus,

$$U_{16,000} = 0.5\ U_{-10,000} + 0.5\ U_{50,000} = 5 \text{ utils.}$$

Next we vary the probabilities in the bet. If the decision-maker is offered a bet involving a loss of Rs 10,000 with a probability, say 1/10 and a gain with a probability 9/10, or, alternatively, a gift of Rs 40,000, and if he prefers the sure award of Rs 40,000 then it is reckoned that the utility of this amount is greater than the expected utility of the bet. Then we keep on lowering down the amount of sure gift to arrive at a value for which he would be indifferent. If this amount is Rs 36,000, we have

$$\begin{aligned} U_{36,000} &= 0.1\ U_{-10,000} + 0.9\ U_{50,000} \\ &= 0.1 \times 0 + 0.9 \times 10 = 9 \text{ utils} \end{aligned}$$

In a similar manner, the different probability values for the reference points may be taken and the particular amounts at which the decision-maker would be indifferent to each of the values determined. The utility values for each of the amounts are then obtained in the same manner as discussed above.

Once the utility values for several amounts of money are derived, they are plotted on the graph as shown in Fig. 13.3.

The amounts of money are shown on the X-axis while the utility measure is depicted vertically. The pairs of money amount and corresponding utility value are plotted on the graph and then they are joined by a *continuous curve*. This is the utility function.

Notice from this function that $U_{16000} = 5$ utils, and $(4/9)\ U_{-10000} + (5/9) U_{36000} = (4/9)0 + (5/9)9 = 5$ utils. As a check on the utility function derived, the decision-maker should be indifferent between a sure amount of Rs 16,000 and a bet involving a loss of Rs 10,000 with a probability 4/9 and a gain of Rs 36,000 with a probability equal to 5/9. If he is not, it implies that the individual in question is not exhibiting consistency in his preferences and the assumptions stated previously are not being met with, and so the utility function needs revision. In case the individual is consistent, the utility function can be used for decision-making.

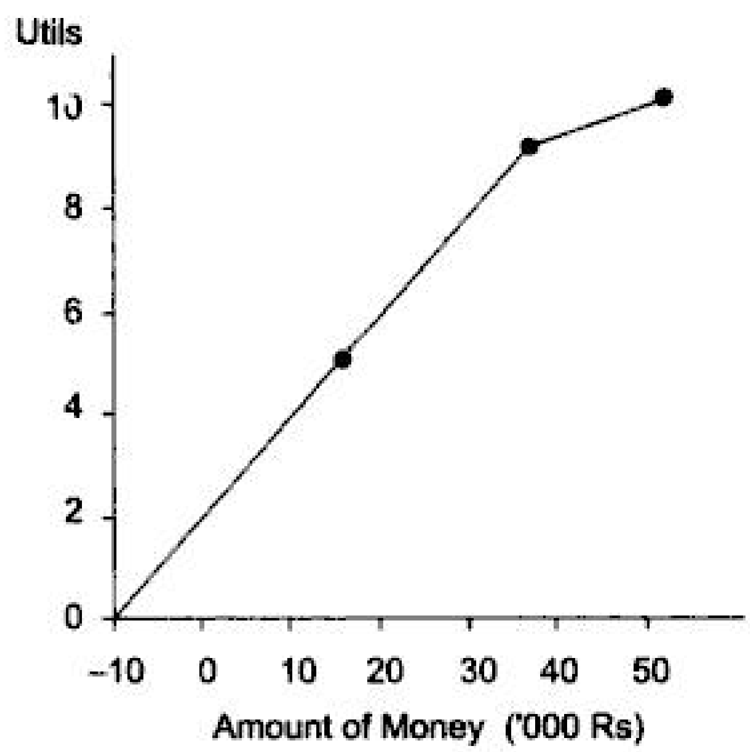


Fig. 13.3 Utility Function

Table 13.28 *Pay-off Table*

Event	Act, A_j	
	A_1	A_2
E_1	1,000	4,000
E_2	1,000	0

(c) If the decision rule to be used is that of minimising the expected cost, the optimal course of action would depend upon the likelihood of each event. Let p be the probability of E_1, so that

$$P(E_1) = p \quad \text{and} \quad P(E_2) = 1 - p.$$

The expected values for each of the acts would be as follows:

$$E(A_1) = 1{,}000p + 1{,}000(1 - p) = 1{,}000$$

$$E(A_2) = 4{,}000p + 0(1 - p) = 4{,}000p.$$

For indifference,

$$1{,}000 = 4{,}000p \quad \text{or} \quad p = 1{,}000/4{,}000 = 0.25.$$

Thus, if the probability that the process is out of control exceeds 0.25, then the variance calls for investigation.

(d) When the process is under control, the variances in the cost will follow normal distribution (as given) with mean = 0 and a standard deviation = Rs 740. The probability of observing a variance of upto Rs 1,200 can be determined as follows:

$$z = \frac{1{,}200 - 0}{740} = 1.62$$

Area beyond $z = 1.62$ equals 0.053. Thus, the probability that a cost variance as great as Rs 1,200 (favourable or adverse) will occur = 2 × 0.053 = 0.11 or 11%. It suggests that the variance should be investigated as the low probability of 11% is indicative of the state that the process might be out of control or at least that the probability of its being out of control is greater than 0.25.

Note here the probability value 0.11 should not be taken to mean that it represents the conditional probability of observing a variance in excess of Rs 1,200 *assuming* that the process is in control. If we denote the event process in control by C and 'process not in control' by C, we have,

$$P(\text{var.} = 1{,}200/C) = 0.11.$$

To obtain the probability that the process is in control having observed the variance of Rs 1,200, we need to use the Bayes' Theorem as follows:

$$P(C/\text{var.} = 1{,}200) = \frac{P(\text{var.} = 1{,}200/C) \times P(C)}{P(\text{var.} = 1{,}200/C) \times P(C) + P(\text{var.} = 1{,}200/\bar{C}) \times P(\bar{C})}$$

However, it is possible to calculate this probability only if we knew $P(C)$ and $P(\text{var.} = 1{,}200/\bar{C})$.

The probabilities of various events are:

$E_1 : 0.7 \times 0.4 = 0.28;\ E_2 : 0.7 \times 0.6 = 0.42;\ E_3 : 0.3 \times 0.4 = 0.12$, and $E_4 : 0.3 \times 0.6 = 0.18$.

The conditional pay-offs, resulting from different combinations of actions and events are given in Table 13.31. Since the expected value for the act A_4 is the largest, it represents the optimal choice.

Table 13.31 *Calculation of Expected Pay-offs*

Event, E_i	Prob.	Act, A_j				
		A_1	A_2	A_3	A_4	A_5
E_1	0.28	0	3,000	5,000	8,000	8,000
E_2	0.42	0	3,000	(2,000)	1,000	(2,000)
E_3	0.12	0	(2,000)	5,000	(2,000)	3,000
E_4	0.18	0	(2,000)	(2,000)	(2,000)	(2,000)
Expected Pay-off		0	1,500	800	2,060	1,400

Example 13.24 Mr X is considering whether to make an investment in a project with the following likely returns:

Amount (Rs)	*Probability*
2,00,000	0.6
–40,000	0.4

The utility function of Mr X is approximated as follows:

$$U = -0.0003M^2 \quad \text{when} \quad M < -5{,}000$$
$$= 1.05M \quad \text{when} \quad M \geq -5{,}000$$

Should the project be undertaken by him? Consider (a) Expected monetary value criterion, and (b) Expected utility criterion.

Calculation of expected monetary value (EMV) and expected utility (EU) is shown in Table 13.32.

Table 13.32 *Calculation of EMV and EU*

Conditional Monetary Value (i)	Conditional Utility* (ii)	Probability (iii)	Expected Monetary Value (i) × (iii)	Expected Utility (ii) × (iii)
200,000	210,000	0.6	120,000	126,000
– 40,000	– 480,000	0.4	–16,000	–192,000
Total			104,000	–66,000

* Obtained by substituting monetary values in the utility function.

Since EMV is positive, the project should be undertaken according to the EMV criterion, while it should not be accepted on the basis of the EU criterion since EU is negative.

5. (a) From the following matrix, the elements of which indicate profits, obtain the decisions using the following principles of decision making:

(i) Maximax (ii) Maximin (iii) Laplace

	a_1	a_2	a_3	a_4	a_5
S_1	26	22	13	22	18
S_2	26	22	34	30	20
S_3	18	22	18	18	20
S_4	22	22	18	18	18

(b) How would your answer in (a) be different if the elements of the matrix represented costs?

6. Find the best alternative in the following decision table using expected value criterion and the criterion of expected regret. The entries in the matrix indicate profits associated with different combinations of acts and states of nature.

State of nature (θ)	Prob.	Act a_1	Act a_2	Act a_3
θ_1	0.6	0	3	–3
θ_2	0.1	2	2	2
θ_3	0.2	5	–1	3
θ_4	0.1	–4	–3	1

7. A dealer in second-hand machinery is offered 5 machines by a company for Rs 5,000 only. He expects to sell each of the machines for Rs 2,200 at a fair but he also knows that any machines not sold would be a waste and not fetch anything later on. Obtain the pay-off matrix under each of the following assumptions:

(a) that the dealer has the option of buying the entire lot only, and

(b) that the dealer has the option of buying as many machines as he chooses, at the rate of Rs 1,000 per machine.

Using expectation principle of choice, determine the optimal strategy for the dealer in each of the two cases given.

8. Three types of souvenirs can be sold outside a stadium. From the following conditional pay-off table, construct the opportunity loss table. (Sales are dependent on the winning team.)

	Types of souvenir		
	I (Rs)	II (Rs)	III (Rs)
Team *A* wins	1,200	800	300
Team *B* wins	250	700	1,100

Point out which type of souvenir should be bought if probability of Team *A*'s winning is 0.6.

(MCom, Delhi, 1994)

9. Chemical Products Ltd produces a compound which must be sold within the month it is produced, if the normal price of Rs 100 per drum is to be obtained. Anything unsold in that month is sold in a different market for Rs 20 per drum. The variable cost is Rs 55 per drum.

The research firm described the reliability of their estimates by means of the following table:

True	*Survey Estimate*	
Event	*Low Demand*	*High Demand*
Low Demand	0.80	0.20
High Demand	0.10	0.90

Should Mr Prime spend money on research to rely on his judgement? What capacity plant should he install? *(MMS Mumbai, 2001)*

31. A company receives shipments of certain items. It should decide whether to accept or reject the shipment, on the basis of inspection of a sample selected from the shipment. From the past experience, it is known that the percentage of defective items in a batch of shipment is either 1, 2 or 5, the probabilities for which are 0.5, 0.3 and 0.2, respectively. The company can accept only those batches which have 1 per cent defectives. The cost of rejecting a good batch, that is, batch with 1 per cent defectives is Rs 600. The cost of accepting a batch with 2 per cent defectives is Rs 400, and the cost of accepting a batch with 5% defectives is Rs 600.

    A sample of 10 items has been selected from the shipment and two items are found to be defective. The conditional probabilities of getting 2 defectives in a sample of 10 items from a batch of 1%, 2% and 5% defective and calculated as 0.083, 0.185, and 0.265 respectively.

    Determine whether the shipment should be accepted.

32. A company has just received a big lot of electronic components for inspection. Experience indicates that there are only two possible compositions of the lot: D_1 5% defective, or D_2 10% defective. Empirical evidence also tells that $P(D_1) = 0.7$ and $P(D_2) = 0.3$. The pay-off matrix for the decision problem is given here (in thousands of rupees of cost of error). These are the only relevant costs.

	D_1	D_2
Accept the lot	0	5
Reject the lot	3	0

    (a) Should the lot be accepted if there is no additional evidence?
    (b) Should the lot be accepted if a sample of two units is drawn and both are defective?
    (c) What is the expected value of the information provided by a sample of two units?
    (d) If testing costs Rs 40 per unit tested, should there be any testing or should lots just be accepted?

33. A company is contemplating the introduction of a new product and is considering the advisability of test marketing done before finally deciding on it. The options on introduction are: to introduce (call it A_1) and not to introduce (call it A_2). The management has estimated the following outcomes and the corresponding prior probabilities.

Outcome	*Profit (Loss)*	*Prob.*
d_1 : the product captures 20% of the market	Rs 10 m	0.70
d_2 : the product captures 7% of the market	Rs 1 m	0.10
d_3 : the product captures 2% of the market	(Rs 5 m)	0.20

of testing is Rs 2,000, the cost of accepting a defective part Rs 20,000 while the cost of rejecting a non-defective part Rs 2,000. Each lot consists of 50 parts and it is assumed that the 10% or 4% defective lots have 3:2 chances of occurrence.

(a) Draw an appropriate decision tree.

(b) State the optimal course of action for the firm indicating whether it is worthwhile to take a sample or not.

(c) Determine the cost of sampling that will make the firm in question neutral between taking and not taking a sample.

(d) Calculate the expected value of perfect information.

41. The utility function of a manager is as follows:

Amount (Rs)	Utility index
–120,000	0.00
–100,000	0.20
–60,000	0.30
0	0.55
40,000	0.62
100,000	0.72
200,000	1.00

The manager is offered a contract which promises a net gain of Rs 1,00,000 or a net loss of Rs 60,000 with equal probabilities.

(a) Should he accept the contract? Use (i) expected monetary value criterion, (ii) expected utility criterion.

(b) Would his decision in (a) above be altered if he is offered to accept the contract twice?

42. Utility function for money of some decision maker is as follows:

Money (Rs)	(–) 1,000	0	5,000	20,000	30,000
Utility	(–) 2	0	10	20	30

A preference test given to him shows that he is indifferent between an investment that will yield Rs 10,000 certainly and a risky venture with a 50% chance of Rs 30,000 profit and a 50% chance of a loss of Rs 1,000. A new risky venture is proposed with possible pay-offs of either Rs 0 or Rs 20,000. The probabilities of gain cannot be determined. Find the probability combination of Rs 0 and Rs 20,000 which will yield Rs 10,000 certainty. *(CA, November, 1994)*

43. A manager has following utility function depicting her attitude towards uncertainty:

Amount (x) Rs	Utility U(x)
0	0.00
1,000	0.25
2,000	0.40
3,000	0.52
4,000	0.68

(ii) The Initial Conditions The initial conditions describe the situation the system presently is in. For instance, as indicated earlier, as the market is divided 30%, 45% and 25% between the brands D_1, D_2 and D_3 respectively on March 1, the current date, it describes the initial condition. It may be expressed in terms of a row vector [0.30 0.45 0.25]. In case the initial condition is described as [0 0 1] for the *market*, it implies that the brand D_3 holds the entire market, while for a *customer*, it would imply that the given customer has currently bought brand D_3. Further, for the gambler's ruin problem, the initial condition is given by [0 0 1 0 0], which implies that he currently is in the state where his capital is Rs 20.

14.4.2 Output

As stated earlier, there are two predictions which a Markov analysis provides. The first of these is the probability of the system being in a particular state at a future time, while the other is the steady state probabilities.

1. Specific-state Probabilities For calculating the probabilities for the system in specific states, we let $q_i(k)$ to represent the probability (q) of the system being in a certain state (i) in a certain period (k), called the state probability. Since the system would occupy one and only one state at a given point in time, it is obvious that the sum of all q_i values would be equal to 1. In general terms, with a total of n states,

$$q_1(k) + q_2(k) + q_3(k) + \dots + q_n(k) = 1, \text{ for every } k$$

in which k is the number of transitions (0, 1, 2, ...)

Let us consider the calculation of the $q_{i(k)}$ probabilities for the detergent example. With states of the system designated as D_1, D_2 and D_3, q_{D1} (0) represents the probability of a customer choosing brand D_1 this month (at $t = 0$) and q_{D1} (1) represents the probability of choosing this brand after one transition, that is, the next month. Similarly, q_{D1} (2) is the probability of choosing this brand after two transitions (in the next to next month) and so on. Using these symbols, the probability distribution of the customer choosing any given brand (D_1, D_2, D_3) in any given month (k) may be expressed as a row vector as follows:

$$Q(k) = [q_{D1}(k)\ q_{D2}(k)\ q_{D3}(k)]$$

In general, for n states

$$Q(k) = [q_1(k)\ q_2(k)\ q_3(k) \cdots q_n(k)]$$

The initial condition is obviously expressed as $Q(0)$.

For the detergent example, since the market share for the three brands D_1, D_2, and D_3 initially (on March 1) is given to be 30%, 45% and 25%, respectively, we can write initial state probabilities as

$$Q(0) = [q_{D1}(0)\ q_{D2}(0)\ q_{D3}(0)] = [0.30\ 0.45\ 0.25]$$

Now, the managers of the three brands of detergents would benefit from knowing the market shares that would occur at a given future time (k). This information would be given by $q(k)$ where $k = 1, 2, 3, \dots$ and so on. To be specific, let us calculate the share of the market likely to be held by each of the brands on April 1 (since the time period considered by us is one month, as mentioned earlier). This would be represented by $Q(1)$, since $k = 1$ for the next month. For this purpose, we use the matrix of transition probabilities, P. The row vector $Q(0)$ would be post-multiplied by the matrix P to get $Q(1)$. Thus,

$$Q(1) = (0.30\ 0.45\ 0.25)\begin{pmatrix} 0.60 & 0.30 & 0.10 \\ 0.20 & 0.50 & 0.30 \\ 0.15 & 0.05 & 0.80 \end{pmatrix} = (0.3075 \quad 0.3275 \quad 0.3650)$$

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Chapter 16

Dynamic Programming

16.1 INTRODUCTION

Dynamic Programming is a useful quantitative analysis technique that can be used to solve many optimisation problems. It deals with relatively large and complex problems and involves making a sequence of interrelated decisions. The technique provides a systematic procedure for determining optimal combination of decisions.

A given problem may be complex if it is not possible to solve it in one go. It may be solved by breaking it into a series of smaller and more tractable sub-problems. This is exactly what dynamic programming does. In most applications, it obtains solutions by working backward from the end of the problem toward the beginning. For this, it divides the given problem into a number of decision stages; the outcome of a decision at a given stage affects the decision at each of the following stages. Solving problems with dynamic programming involves the following four steps:

1. Divide the original problem into sub-problems, called stages.
2. Solve the last stage of the problem for all possible conditions or states.
3. Moving backward from the last stage, solve each intermediate stage successively. This is done by determining optimal policies from that stage to the end of the problem.
4. Obtain the optimal solution to the original problem by solving all stages sequentially.

alternative routes to reach stage 4, namely 4-7 and 4-8. The first of these requires 27 hours plus the optimal time to reach city *B* from state 7 in stage 4, equal to 21 hours, making a total of 48 hours. Similarly, the second of the routes, 4–8, needs 30 hours plus 15 hours from state 8 in stage 4 to the destination city, equaling 45 hours.

The conditional payoffs in respect of each of the states in stage 3 are indicated in Table 16.3.

Table 16.3 *Payoffs at Stage 3*

State	*Alternative Routes*	*Time (hrs)*			*Best Route*
		To stage 4	*from stage 4 to city B*	*Total*	
4	4–7	27	21	48	
	4–8	30	15	45	√
5	5–8	24	15	39	√
	5–9	30	18	48	
6	6–9	24	18	42	√
	6–10	21	27	48	

In terms of Table 16.3, the following are the conditional optimal decisions:

(a) if the traveler is at exchange point 4, he would travel 4-8-11,
(b) if he is at exchange point 5, he would go 5-8-11, and
(c) in case the traveler is at point 6, the best route to follow is 6-9-11.

Stage 2 In order to get to either exchange point 4, 5 or 6 in stage 3, the traveler has to be at exchange point 2 or 3–the states of stage 2. Here also, we seek to find the optimal route from each of these points to City *B*. But, instead of enumerating all routes to the destination city, only the routes to the states of stage 3 are examined and the results obtained in stage 3 serve as input for computation in stage 2. The computations are shown in Table 16.4.

Table 16.4 *Payoffs at Stage 2*

State	*Alternative Routes*	*Time (hrs)*			*Best Route*
		To stage 3	*From stage 3 to city B*	*Total*	
2	2–4	18	45	63	
	2–5	15	39	54	√
3	3–5	12	39	51	√
	3–6	18	42	60	

Stage 1 Now the traveler is at the initial point, City A, where he has two alternatives: either go to exchange point 2 or exchange point 3. There is only one state to examine here. The results are given in Table 16.5.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

$$\sigma = \sqrt{\sum_{j=1}^{n} \left[\text{NPV}_j - E(\text{NPV})\right]^2 p_j}$$

NPV_j = NPV of the *j*th combination,

p_j = probability of the *j*th combination

$\sigma = [(-11{,}526 - 20{,}177)^2 \times 0.08 + (-2{,}953 - 20{,}177)^2 \times 0.08 + \ldots]^{1/2}$ = Rs 18,684.

Although useful for setting out all possible combinations of a proposed project, the decision-tree approach suffers from a shortcoming that in situations involving a large number of possible outcomes, it may be too complex to handle.

18.5.5 Simulation Approach to Risky Investments

Another technique by which risk in investment may be considered is the simulation technique that was discussed in Chapter 17. Simulation can be used to approximate the NPV or the expected return and its dispersion, about the expected value. For its application, in the first stage, the various factors influencing the cash flows relating to a project are identified. Then the probability distributions of each of them are obtained, either using the historical data, if available, or the decision maker's judgment and expert knowledge. In the next stage, values for each of the variables are simulated in accordance with the procedure laid in the Chapter 17, and the NPV, or the expected return, resulting from a combination thereof is computed. This procedure is repeated a large number of times and the probability distribution of the NPV (or the rate of return) is obtained. This distribution is then used as a basis for taking appropriate decisions relating to the proposal.

A simulation model proposed by Hertz* considers the following factors in analysing risky investment projects:

Market Analysis:

1. Market size
2. Selling price
3. Market growth rate
4. Share of market (which results in physical sales volume)

Investment Cost Analysis:

1. Investment required
2. Residual value of investment

Operating and Fixed Cost:

1. Operating costs
2. Fixed costs
3. Useful life of facilities.

As mentioned earlier, probability distributions are assigned to each of these factors, on the basis of the management's assessment of the probable outcomes. Once this is done, the next present value or the rate of return can be obtained that will result from a random combination of the nine factors listed.

---

* David B. Hertz, "Risk Analysis in Capital Investment" *Harvard Business Review*, 42 (Jan-Feb 1964) pp. 95-106.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.

Log on to http://www.mhhe.com/vohra3e

QUANTITATIVE TECHNIQUES IN
Management

THIRD EDITION

This well-known text provides a conceptual understanding of various quantitative techniques used in solving management problems.

The main objectives of the new edition are: to expand the coverage by including a chapter on Dynamic Programming; to include more exercises on the topic Goal Programming; to enhance the coverage of PERT/CPM; to refine the treatment of Inventory Management; and to develop a comprehensive suite of supplements which are offered through the companion website of the book.

Learning Aids

- Chapter *Overview* and *Contents*
- In-text Schematics of algorithms and solved *Examples*
- *Key Points to Remember*: end-of-chapter summary
- *Test Your Understanding*: end-of-chapter objective questions
- *Exercises*: end-of-chapter concept-testing questions
- *Practical Problems*: end-of-chapter numerical problems (with answers at the end of the book)

N D Vohra is Reader, Department of Commerce, Ramjas College, University of Delhi. He has a wide teaching experience including that at the Delhi School of Economics, The Institute of Chartered Accountants of India and other institutions offering professional courses.

The McGraw-Hill Companies

ISBN-13: 978-0-07-061193-1
ISBN-10: 0-07-061193-9

9780070611931

Visit Tata McGraw-Hill at:
www.tatamcgrawhill.com